POPA SUPERRIGIDITY AND INCOMPARABLE ACTIONS OF LINEAR GROUPS

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ABSTRACT. We prove strong ergodicity results for the shift action of linear algebraic groups, restricted to the free part. This yields to various incomparability results for countable Borel equivalence relations including the first result of continuum many pair-wise incomparables of Adams and Kechris. Our proof makes use of Popa’s cocycle superrigidity theorem and avoids the machinery of Zimmer’s cocycle superrigidity.

1. INTRODUCTION

A breakthrough result in the theory of countable Borel equivalence relations (CBERs) was the existence of continuum many CBERs pair-wise incomparable countable Borel bi-reducibility. This was proved by Adams and Kechris [1], who built on the work of Zimmer and developed a whole arsenal of techniques to analyze action of linear groups.

Following the approach of Thomas [7], we analyze a large class of equivalence relations induced by the shift action of linear group on the space of their subsets. We give a slicker proof of Adams and Kechris’s original result [1, Lemma 4.5], and we extend it to higher (and lower) dimensions.

Let \( P = \{ p \text{ prime} \mid p \text{ is odd} \} \). Let for each nonempty set of primes \( S \subseteq P \), let \( \text{SO}_n(\mathbb{Z}[S^{-1}]) \) be the group of \( n \times n \) orthogonal matrices with determinant 1 with coefficients in the ring \( \mathbb{Z}[S^{-1}] \) of rationals whose denominators, in reduced form, have prime factors in \( S \). Whenever \( S = \{ p_1, \ldots, p_n \} \) is a finite set, we write \( \text{SO}_n(\mathbb{Z}[\frac{1}{p_1}, \ldots, \frac{1}{p_n}]) \) instead of \( \text{SO}_n(\mathbb{Z}[S^{-1}]) \). For a countable group \( G \) let \( F(G, 2) \) denote the countable Borel equivalence relation induced by the shift action of \( G \) on the free part.

**Theorem 1.1.** Let \( n \geq 5 \) and \( S, T \subseteq P \). Then,

\[
S \subseteq T \iff F(\text{SO}_n(\mathbb{Z}[S^{-1}]), 2) \leq_B F(\text{SO}_n(\mathbb{Z}[T^{-1}]), 2).
\]
Theorem 1.2. Let \( n \geq 5 \) and \( S \subseteq \mathbb{P} \). Then,
\[
m \leq n \iff F(\text{SO}_n(\mathbb{Z}[S^{-1}], 2)) \leq_B F(\text{SO}_n(\mathbb{Q}), 2).
\]

Remark 1.3. For \( n \geq 6 \) we can replace \( \mathbb{P} \) with the set of all prime numbers. We do not know if \( \text{SO}_5(\mathbb{Z}[\frac{1}{2}]) \) has Kazhdan's property (T). If that is the case, one can replace \( \mathbb{P} \) with the set of prime numbers in the statements of Theorem 1.1 and Theorem 1.2.

Throughout this paper, we assume familiarity with the basic notions of algebraic groups. A brief account of our terminology can be found in [4, Section 2.2] (or [8, Chapter 3]).

2. Preliminaries

Let \( E, F \) be countable Borel equivalence relations on the standard Borel spaces \( X, Y \) respectively. Then a Borel map \( f : X \rightarrow Y \) is said to be a homomorphism from \( E \) to \( F \) iff for all \( x, y \in X \),
\[
x E y \implies f(x) F f(y).
\]
If \( f \) satisfies the stronger property that for all \( x, y \in X \),
\[
x E y \iff f(x) F f(y),
\]
then \( f \) is said to be a Borel reduction and we write \( E \leq_B F \).

Most of the Borel equivalence relations that we will consider in this paper arise from group actions as follows. Let \( G \) be a countable discrete group. Then a standard Borel \( G \)-space is a standard Borel space \( X \) equipped with a Borel action \( G \times X \rightarrow X, (g, x) \mapsto g \cdot x \) of \( G \) on \( X \). We denote by \( \mathcal{R}(G \actson X) \) the corresponding orbit equivalence relation on \( X \), whose classes are the \( G \)-orbits. That is,
\[
\mathcal{R}(G \actson X) := \{(x, y) \in X^2 \mid \exists g \in G (g \cdot x = y)\}.
\]

For any infinite countable group \( G \), let \( 2^G \) be the compact Polish space of all functions from \( G \) into \( 2 = \{0, 1\} \) with the product topology. Any countable group \( G \) acts on \( 2^G \) by (left) shift: for any \( g \in G \) and \( f \in 2^G \), let \( g \cdot f(x) = f(g^{-1}x) \). Hence, we regard \( 2^G \) as a standard Borel \( G \)-space, and we denote by \( E(G, 2) \) the corresponding equivalence relation. The free part of the shift action \( G \actson 2^G \), is defined as the set
\[
(2)^G := \{x \in X \mid \forall g \in G (g \neq 1 \implies g \cdot x \neq x)\}.
\]
We let \( F(G, 2) \) be the restriction of \( E(G, 2) \) to the free part of the shift action, i.e., \( F(G, 2) = \mathcal{R}(G \actson (2)^G) \).

The forward implications of Theorem 1.1 and Theorem 1.2 follow from the following well-known fact.

Proposition 2.1. If \( G \) embeds into \( H \), then \( F(G, 2) \leq_B F(H, 2) \).
Proof. The map \((2)^G \to (2)^H, f \mapsto f^*\) with
\[
f^*(x) = \begin{cases} f(x) & \text{if } x \in G, \\ 0 & \text{otherwise.} \end{cases}
\]
is a Borel reduction. \(\square\)

Let \(G\) be a countably infinite group and let \(X\) be a standard Borel \(G\)-space. Throughout this paper, a probability measure on \(X\) will always mean a Borel probability measure; i.e., a measure which is defined on the collection of Borel subsets of \(X\). The probability measure \(\mu\) on \(X\) is \(G\)-invariant if and only if \(\mu(g(A)) = \mu(A)\) for every \(g \in G\) and Borel subset \(A \subseteq X\). If \(\mu\) is \(G\)-invariant, then the action of \(G\) on \((X, \mu)\) is said to be ergodic if and only if for every \(G\)-invariant Borel subset \(A \subseteq X\), either \(\mu(A) = 0\) or \(\mu(A) = 1\). In this case, we say that \(\mu\) is an ergodic probability measure. The following characterization of ergodicity is well-known. (E.g., see [3].)

Proposition 2.2. If \(\mu\) is a \(G\)-invariant probability measure on the standard Borel \(G\)-space \(X\), then the following statements are equivalent.

(i) The action of \(G\) on \((X, \mu)\) is ergodic.

(ii) If \(Y\) is a standard Borel space and \(f : X \to Y\) is a \(G\)-invariant Borel map, then there exists a Borel subset \(M \subseteq X\) with \(\mu(M) = 1\) such that \(f \restriction M\) is constant.

Suppose that \(E\) is a countable Borel equivalence relation on the standard Borel space \(X\) and that \(\mu\) is a probability measure on \(X\). Then \(\mu\) is said to be \(E\)-invariant if \(\mu\) is \(G\)-invariant for some (equivalently every) countable group \(G\) with a Borel action on \(X\) such that \(E = \mathcal{R}(G \curvearrowright X)\).

We denote by \(\mu_G\) the product measure on \(2^G\) (where each bit in \(\{0,1\}\) has measure \(1/2\)). Clearly, \(\mu_G\) is invariant under the shift action and it is non-atomic. For any infinite countable group \(G\), we have \(\mu_G((2)^G) = 1\). In the next section we will be using the following fact. (For a proof see [1, Lemma 4.4])

Fact 2.3. Let \(H \leq G\) be a countable infinite subgroup. Then the shift action of \(H\) on \(((2)^G, \mu_G)\) is ergodic.

In this paper, we consider the following strong form of ergodicity.

Definition 2.4. If \(E\) is a Borel equivalence relation on a standard Borel space \(X\), \(\mu\) is a probability measure on \(X\) and \(F\) is a Borel equivalence relation on a standard Borel space \(Y\), we say that \(E\) is \(F\)-ergodic if for any Borel homomorphism \(f : X \to Y\) from \(E\) to \(F\), there is a Borel \(E\)-invariant set \(M \subseteq X\), with \(\mu(A) = 1\), such that \(f\) maps \(A\) into a single \(F\)-class.

If \(E\) and \(F\) are countable Borel equivalence relation, \(\mu\) is a non-atomic probability measure and \(E\) is \(F\)-ergodic then there is no countable-to-one Borel homomorphism from \(E\) to \(F\). In particular, it follows that \(E \not\leq_B F\).
3. Strong ergodicity

First, we recall the notion of Borel cocycle and state a useful consequence of Popa’s superrigidity theorem. Suppose that $\Gamma$ is a standard Borel space with an invariant probability measure $\mu$ and $H$ is a countable group.

A Borel function $\alpha : G \times X \to H$ is a cocycle if for all $g, h \in G$,

$$\alpha(hg, x) = \alpha(h, g \cdot x)\alpha(g, x) \quad \text{for } \mu\text{-a.e. } x \in X.$$  

The cocycles $\alpha, \beta : G \times X \to H$ are equivalent if there exists a Borel map $B : X \to H$ such that for all $g \in G$,

$$\beta(g, x) = B(g \cdot x)\alpha(g, x)B(x)^{-1} \quad \text{for } \mu\text{-a.e. } x \in X.$$  

The strong ergodicity results in this section follows from the following special case of Popa’s cocycle superrigidity theorem isolated in Thomas [7].

**Theorem 3.1** (Popa [6]). Let $\Gamma$ be a countably infinite group with property (T) and let $G, H$ be countable groups such that $\Gamma \leq G \leq \mathcal{G}$. If $H$ is any countable group, then every Borel cocycle

$$\alpha : G \times (2)^G \to H$$  

is equivalent to a group homomorphism of $G$ into $H$.

Let $G$ be an algebraic group over $k$. Recall that $G$ is said to be almost $k$-simple if the only proper algebraic normal subgroups of $G$ defined over $k$ are finite. The $k$-rank of $G$ denoted by $\text{rank}_k G$, is the dimension of a maximal $k$-split torus. In the proof of the lemma below we consider the $\mathbb{Q}_p$-rank of $SO_n$ for different $n$ and $p$. For a useful summary about the $\mathbb{Q}_p$-rank of $SO_n$ we refer the reader to [5, Section 4].

**Lemma 3.2.** For each $p \in \mathbb{P}$, the group $SO_n(\mathbb{Z}[1/p])$ has property (T).

**Proof.** We regard $SO_n(\mathbb{Z}[1/p])$ as a subgroup of $SO_n(\mathbb{R}) \times SO_n(\mathbb{Q}_p)$ under the diagonal embedding. The image of $SO_n(\mathbb{Z}[1/p])$ under the projection $\pi : SO_n(\mathbb{R}) \times SO_n(\mathbb{Q}_p) \to SO_n(\mathbb{Q}_p)$ is a lattice in $SO_n(\mathbb{Q}_p)$ because $\ker \pi = SO_n(\mathbb{R})$ is compact. Recall that $SO_n$ is almost $\mathbb{Q}_p$-simple for $n \geq 5$. Since $\mathbb{Q}_p$-rank $SO_n \geq 2$, it follows that $SO_n(\mathbb{Q}_p)$ has property (T) by [2, Theorem 1.6.1]. Therefore, $SO_n(\mathbb{Z}[1/p])$ has property (T). \[\square\]

Next lemma shows that all group homomorphisms between the linear groups considered here are “almost trivial”.

**Lemma 3.3.** Let $n \geq 5$. If $p \not\in T$, then every group homomorphism $\rho : SO_n(\mathbb{Z}[1/p]) \to SO_n(\mathbb{Z}[T^{-1}])$ has finite image.

**Proof.** Let $G$ be the universal cover of $SO_n$ and $\pi : G \to SO_n$ be usual the 2-to-one covering map. Put $\Gamma = \pi^{-1}(SO_n(\mathbb{Z}[1/p]))$. Clearly, it suffices to show that every group homomorphism $\rho : \Gamma \to SO_n(\mathbb{Z}[T^{-1}])$ has finite image. For every prime $p \not\in T$, the group $SO_n(\mathbb{Z}[1/p])$ has property (T).
number $q \in T$, let $Q_q$ be the field of $q$-adic numbers. And let $SO_m(Q) \to SO_m(Q_q)$ be the canonical embedding so that we can view $\rho$ as a map $\rho_q : \Gamma \to SO_m(Q_q)$.

Since $\Gamma$ is finitely generated, there are only finitely many prime numbers $p_1, \ldots, p_n$ that appear in the denominators of the entries of the matrices in $\varphi(\Gamma)$. So, it suffices to show that the power of each prime appearing in the denominators of the matrixes in $\rho(\Gamma)$ is bounded. Equivalently, we can show that for every $q \in \{p_1, \ldots, p_n\}$, let $\rho(\Gamma)$ is relatively compact in $SO_m(Q_q)$. This follows from the Margulis’ superrigidity theorem (e.g., see [9, Theorem 10.1.5]) arguing exactly as in [4, Lemma 4.16].

A similar argument yields the following analogue of Lemma 3.3.

**Lemma 3.4.** Let $n \geq 5$. If $p \in \mathbb{P}$ and $m < n$, then every group homomorphism $\rho : SO_n(\mathbb{Z}[\frac{1}{p}]) \to SO_m(Q)$ has finite image.

**Theorem 3.5.** Let $n \geq 5$. If $S \not\subseteq T$, then $F(SO_n(\mathbb{Z}[S^{-1}]), 2)$ is $F(SO_n(\mathbb{Z}[T^{-1}]), 2)$-ergodic.

**Proof.** For short let $G = F(SO_n(\mathbb{Z}[S^{-1}]), 2)$ and $H = SO_n(\mathbb{Z}[T^{-1}])$. Suppose that $f$ is a Borel reduction from $F(G, 2)$ is $F(H, 2)$. Let $p \in S \setminus T$ and put $\Gamma = SO_n(\mathbb{Z}[\frac{1}{p}])$. Then $f$ is a countable-to-one Borel homomorphism from $\mathcal{R}(\Gamma \bowtie (\mathbb{Z}[S^{-1}]), 2)$ to $F(SO_n(\mathbb{Z}[T^{-1}]), 2)$. Let $\alpha : \Gamma \times (\mathbb{Z}[S^{-1}]) \to H$ be the cocycle associated to $f$. That is, we let $\alpha(g, x) \in H$ be the unique group element such that $f(g \cdot x) = \alpha(g, x) \cdot f(x)$.

By Theorem 3.1 there are a group homomorphism $\rho : \mathcal{R}(SO_n(\mathbb{Z}[\frac{1}{p}])) \to SO_n(\mathbb{Z}[T^{-1}])$, a Borel function $B : X \to SO_n(\mathbb{Z}[T^{-1}])$, and $X_0 \subseteq X$ with $\mu_C(X_0) = 1$ such that

$$\rho(g) = B(g \cdot x)\alpha(g, x)B(x)^{-1},$$

for all $g \in \Gamma$ and $x \in X_0$. It is straightforward to check that $X_0$ is $\mathcal{R}(\Gamma \bowtie X)$ is invariant. That is, if $x \in X_0$ and $(x, y) \in \mathcal{R}(\Gamma \bowtie X)$, then $y \in X_0$.

Then define $f' : X \to Y$ by setting $f'(x) = B(x) \cdot f(x)$, so that $\rho$ is the cocycle associated to $f'$. That is, $f'(g \cdot x) = \rho(g) \cdot f'(x)$ for all $g \in \Gamma$, $x \in X$. Now, for all $x \in X_0$, let

$$\Phi(x) = \{\rho(g) \cdot f'(x) \mid g \in G\}$$

$$= \{f'(z) \mid (z, x) \in \mathcal{R}(G \bowtie X)\}.$$

For each $x \in X_0$, it is clear that $\Phi(x)$ is a nonempty finite subset of $Y$ and if $(x, y) \in \mathcal{R}(\Gamma \bowtie X)$ then $\Phi(x) = \Phi(y)$. Therefore, $\Phi$ is a $\Gamma$-invariant Borel map into $\mathcal{P}_f(\Gamma)$. If $(X, \mu)$ is ergodic, then there is a $\Gamma$-invariant $M \subseteq X_0$ with $\mu(M) = 1$ such that $\Phi \upharpoonright M$ is constant. It follows that $f$ maps $M$ into a single class of $\mathcal{R}(H \bowtie Y)$, which is a contradiction because $f$ is countable-to-one and $\mu_C$ is non-atomic. □

**Theorem 3.6.** Let $n \geq 5$. If $m < n$, then $F(SO_n(\mathbb{Z}[S^{-1}]), 2)$ is $F(SO_m(Q), 2)$-ergodic.
The proof of Theorem 3.6 goes exactly as the one of Theorem 3.5 making the obvious use of Lemma 3.4 instead of Lemma 3.3.

References