Countable Borel equivalence relations

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These notes cover an introduction to countable Borel equivalence relations. The theory of countable Borel equivalence relation is a huge subject. It has a broad impact in several areas of mathematics like ergodic theory, operator algebras, and group theory, and has raised many challenging open problems. We focus on some results about the (anti)classification of torsion-free abelian groups, some applications of Popa’s cocycle superrigidity theorem, and Martin’s conjecture.

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1 Background

Polish spaces

In this section we introduce the central notions of Polish and standard Borel spaces, and we list some fundamental theorems about them. For an extensive exposition on this topics and other fundamental aspects of descriptive set theory we refer the interested reader to Kechris’ book [22] or the excellent lecture notes of Tserunyan [45].

Definition 1.1. A topological space \((X, \tau)\) is said to be Polish if it is separable and metrizable (i.e., there is a complete metric that generates \(\tau\)).

Example 1.2. Among examples of Polish spaces we have the following well-known topological spaces:

- For all \(n \in \mathbb{N}, n = \{0, \ldots, n-1\}\) with the discrete topology is Polish. So is \(\mathbb{N}\) and any countable set.
- \(\mathbb{R}^n, \mathbb{C}^n\) and the \(n\)-dimensional torus \(\mathbb{T}^n\), the unit interval \(I = [0, 1]\) and \(I^n\), for all \(n \in \mathbb{N}\).
- All separable Banach spaces, such as the \(\ell_p\) spaces for \(1 \leq p < \infty\), and \(c_0\), the space of converging to 0 sequences with the sup norm.
- The Cantor space \(2^\mathbb{N}\), viewed as the product of infinitely many copy of \(\{0, 1\}\) with the discrete topology, is a Polish spaces. So is the Baire space \(\mathbb{N}^\mathbb{N}\). In both cases a compatible metric is given by \(d(x, y) = 2^{-(n+1)}\) where \(n\) is the least integer such that \(x(n) \neq y(n)\).

We list a few useful facts abut Polish spaces.

Fact 1.3. Countable product of Polish spaces is Polish with respect to the product topology.

In particular, \(\mathbb{R}^\mathbb{N}\) and \([0, 1]^\mathbb{N}\) are all Polish spaces.
Fact 1.4. A subset of a Polish space is Polish (with respect to relative topology) if and only if it is $G_\delta$ (i.e., countable intersection of open sets).

Notice that Fact 1.4 applies to closed subset, since all closed subsets of any metric space are $G_\delta$.

Theorem 1.5. Every Polish space is homeomorphic to a closed subset of $\mathbb{R}^n$.

Thus, the Polish spaces are, up to homeomorphism, exactly the closed subsets of $\mathbb{R}^n$.

Standard Borel spaces

Recall that a Borel (or measurable) space is a pair $(X, B)$, where $X$ is a set and $B$ is a $\sigma$-algebra on $X$ (i.e., a collection of subsets of $X$ containing $\emptyset$ and closed under complements and countable unions). We refer to the sets in $B$ as Borel sets. For any topological space $(X, \tau)$ we denote by $B(\tau)$ the $\sigma$-algebra generated by $\tau$, which is defined as the smallest subset of $\mathcal{P}(X)$ containing $\tau$.

A common practice throughout this course, when we look at Polish spaces, is to “forget” the topology and consider the induced Borel structure.

Definition 1.6. A Borel space $(X, B)$ is said to be standard if there exists a Polish topology $\tau$ on $X$ that generates $B$ as its Borel $\sigma$-algebra; in which case, we write $B = B(\tau)$. In this case, we call $\tau$ a compatible Polish topology.

An obvious example of a standard Borel space is $(X, B(\tau))$, whenever $(X, \tau)$ is a Polish space.

If $(X, B)$ is a standard Borel space and $Y \subseteq X$ is Borel, then we can regard $Y$ as a Borel space with the relative Borel structure $B \upharpoonright Y = \{B \cap Y : B \in B\} = \{B \in B : B \subseteq Y\}$. It is known that given a Polish space $(X, \tau)$ and $A \in B(\tau)$, there is a Polish topology $\tau_A \supseteq \tau$ such that $A$ is a clopen set in $\tau_A$. (E.g., see Kechris [22, Theorem 13.1].) It follows that the the class of standard Borel spaces is closed under Borel subspaces:

Fact 1.7. If $(X, B)$ is a standard Borel space and $Y \in B$, then $(Y, B \upharpoonright Y)$ is also a standard Borel space.

If $X$ is a topological space, denote by $F(X)$ the set of its closed subset. We endow $F(X)$ with the $\sigma$-algebra $\mathcal{F}$ generated by the sets

$$\{F \in F(X) : F \cap U \neq \emptyset\},$$

where $U$ is an open set of $X$. The measurable space $(F(X), \mathcal{F})$ is called the Effros Borel space of $X$.

Fact 1.8. The Effros Borel space of any Polish topological space is standard.
The Effros Borel spaces have a lot of useful application. For example it can be used to define the (hyper)space of all Polish spaces and the (hyper)space of all separable Banach spaces.

**Example 1.9** (The standard Borel space of all Polish spaces). By Theorem 1.5 every Polish space is (isomorphic to) an element of $F(\mathbb{R}^N)$.

**Example 1.10** (The standard Borel space of all separable Banach spaces). Any separable Banach space is linearly isometric to a closed subspace of $C([0,1])$, the space of continuous functions $[0,1] \to \mathbb{R}$ equipped with the sup norm. Then we define $X_{\text{Ban}}$ as the subspace of $F(C([0,1]))$ given by the Borel subset of all closed linear subspaces of $C([0,1])$ with the relative Borel structure. Each separable Banach space is isomorphic to an element of $X_{\text{Ban}}$.

Let $(X, A)$ and $(Y, B)$ be Borel spaces. A map $f : X \to Y$ is said to be Borel if $f^{-1}(B) \in A$ for each $B \in B$. A Borel isomorphism between $X,Y$ is a bijection $f : X \to Y$ such that both $f$, $f^{-1}$ are Borel.

The following is a consequence of a deep result in descriptive set theory known as Souslin’s theorem.

**Theorem 1.11.** If $X, Y$ are standard Borel spaces and $f : X \to Y$, then the following are equivalent:

1. $f$ is Borel;
2. Graph($f$) $\subseteq X \times Y$ is a Borel set.

Consequently, we obtain the following:

**Theorem 1.12** (Luzin-Suslin). Let $X, Y$ be Polish spaces and $f : X \to Y$ be Borel. If $A \subseteq X$ is Borel and $f \upharpoonright A$ is injective, then $f(A)$ is Borel.

Next theorem shows that the theory of the Borel structure of Poish spaces is very robust.

**Theorem 1.13** (Kuratowski). Any two uncountable Polish spaces are Borel isomorphic.

Thus, standard Borel spaces are fully classified by their cardinality, which can be either countable or $2^{\aleph_0}$.

**Luzin-Novikov uniformization theorem**

A very common misconception about Borel sets is thinking that the image of any Borel set through a Borel function is itself Borel. This is true when the map is assumed to be injective (cf. Theorem 1.12) but it is false in general. For example the projection of a Borel measurable set in the
plane is not necessarily Borel measurable\footnote{This observation is attributed to Suslin, who noticed a mistake in a paper by Lebesgue asserting the contrary. More historical remarks can be found in Moschovakis \cite{Moschovakis}}. In this section we discuss other sufficient conditions for having Borel projection. This is tightly connected with the notion of uniformization and a classical uniformization theorem.

**Definition 1.14.** Given two standard Borel spaces $X, Y$ and $R \subseteq X \times Y$, a *uniformization* of $R$ is a subset $R^* \subseteq R$ such that $R^*$ meets each vertical section $R_x := \{ y \in Y : (x,y) \in R \}$ in exactly one point. (See Figure 1.) Equivalently; $R^* = \text{Graph}(g)$ with $\text{dom } g = \text{proj}_X(R)$ and such that $f(x) \in R_x$, for all $x \in \text{dom}(g)$. In this case, $g$ is called uniformizing function.

If $R \subseteq X \times Y$ are as above, it follows from the axiom of choice that a uniformization for $R$ always exists. It is natural to ask whether $R \subseteq X, Y$ admits a “nice” uniformization. In particular, we are interested in the case when $R$ is Borel and ask whether a Borel uniformization $R^*$ exists. Notice that a positive answer to this question will imply that $\text{proj}_X(R) = \text{proj}_X(R^*)$ is Borel subset of $X$ by Luzin-Suslin theorem.

![Figure 1: Uniformization](image)

Clearly, a uniformization for $R$ can be seen as the graph of a partial function $g$: $\text{dom}(g) \subseteq X \to Y$. More in general, for topological spaces $X, Y$, a set $A \subseteq X \times Y$ is said a *function graph* if $A = \text{Graph}(g)$ for some partial function $g: X \to Y$; equivalently, for every $x \in X$, the *vertical section* $A_x := \{ y \in Y : (x,y) \in R \}$ has at most one element.
Theorem 1.15 (Luzin-Novikov). Let $X, Y$ be Polish spaces and $R \subseteq X \times Y$ be a Borel set all of whose sections $R_x$ are countable. Then $R$ has a Borel uniformization.

Moreover, $R$ can be partitioned into countably many Borel function graphs $R = \bigsqcup_n \text{Graph}(g_n)$.

As an application of Theorem 1.15 we get that the class of Polish spaces is closed under countable-to-one Borel images:

**Exercise 1.16.** Let $X, Y$ be Polish spaces, and $f : X \to Y$ be a countable-to-one Borel function. If $A \subseteq X$ a Borel set, then $f(A)$ is Borel.

We continue with a striking application of Luzin-Novikov’s theorem, which is central in the theory of countable Borel equivalence relations.

Given a standard Borel space $X$, an equivalence relation $E$ on $X$ is said to be Borel if $E$ is a Borel subset of $X \times X$. We call $E$ countable if every $E$-class is countable. In a similar way we call $E$ finite if every $E$-class is finite.

If $G$ is a countable discrete group, then a standard Borel $G$-space is a standard Borel space $X$ equipped with a Borel $G$-action $(g, x) \mapsto g \cdot x$. The induced orbit equivalence relation, denoted by $E_X^G$, is the equivalence relation on $X$ whose classes are the obits of the action.

**Theorem 1.17 (Feldman-Moore [10]).** If $E$ is a countable Borel equivalence relation on the standard Borel space $X$, then there exists a countable group $G$ and a Borel action of $G$ on $X$ such that $E = E_X^G$.

**Proof.** Since $E$ is countable, $E$ is a subset of $X \times X$ with countable sections. It follows from Theorem 1.15 that $E$ can be partitioned into the union of countably many Borel function graphs $\bigsqcup_n \text{Graph}(g_n)$.

Now consider $E^{-1} := \{(x, y) : (y, x) \in E\}$. By applying Luzin-Novikov again we obtain $E^{-1} = E = \bigsqcup_m h_m$. Then $E = \bigsqcup_{m,n} (g_n \cap h_m^{-1})$. So by possibly replacing $(g_n)_n$ with $(g_n \cap h_m^{-1})_{n,m}$, we may assume without loss of generality that each $g_n$ is injective. We may assume without loss of generality that each $g_n$ is injective.

Now we want to extend each $g_n$ to a Borel involution of the whole space. We can do that if $\text{dom}(g_n)$ and $\text{ran}(g_n)$ are disjoint for all $n$. Since $X$ is Hausdorff, the diagonal $\Delta(X)$ is closed. Then by second countability we can write $X^2 \setminus \Delta(X) = \bigcup(U_n \times V_n)$ where $U_n, V_n \subseteq X$ are disjoint open sets. It follows that

$$E = \Delta(X) \cup (E \cap \bigcup_m U_m \times V_m) = \Delta(X) \cup \bigcup_{m,n} (g_n \cap (U_m \times V_m))$$
Then we can extend each $g_n$ to an involution $\tilde{g}_n : X \to X$ by setting

$$\tilde{g}_n(x) = \begin{cases} 
  g_n(x) & x \in \text{dom}(g_n), \\
  g_n^{-1}(x) & x \in \text{ran}(g_n), \\
  x & \text{otherwise}.
\end{cases}$$

We have $E = \bigcup \text{Graph}(g_n)$.

Let $G = \langle \tilde{g}_n \mid n \in \mathbb{N} \rangle$. It follows that

$$x E y \iff \exists n (\tilde{g}_n(x) = y) \iff x E_G y,$$

as desired.

**Remark 1.18.** In fact, every countable Borel equivalence relation $E$ on a Polish space $X$ is a countable union of graphs of Borel involutions on $X$, i.e. $E = \bigcup_{n \in \mathbb{N}} \text{Graph}(g_n)$, where each $g_n : X \to X$ is a Borel involution. In particular, $E$ is the orbit equivalence relation of a Borel action of a countable group generated by involutions.

In this course we are interested in Borel equivalence relation. However, we can adopt a descriptive theoretical approach to analyze other sort of binary relations definable on standard Borel spaces. An interesting case is the one of quasi-orders, i.e., reflexive and transitive binary relations. The systematic study of definable quasi-order began with the fundamental paper of Louveau and Rosendal [29]. A quasi-order $Q$ on $X$ is said to be countable if for all $x_0 \in X$ the set $\{x \in X : x Q x_0\}$ is countable. The following result, which describe countable Borel quasi-order from a dynamical point of view, is the analogue of Feldman-Moore theorem.

**Theorem 1.19** (Williams [48]). If $Q$ is a countable Borel quasi-order on the Polish space $X$, then there is a monoid $M$ acting $X$ in a Borel way such that $x Q y \iff \exists m \in M (x = m \cdot y)$.

**Proof.** Exercise. 

2 Smoothness

**Definition 2.1.** We say that a Borel equivalence relation $E \subseteq X \times X$ is smooth if there exists a Polish space $Y$, and a Borel function $f : X \to Y$ such that $x E y$ holds if and only if $f(x) = f(y)$ for all $x, y \in Y$. 

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An equivalent condition for smoothness is the existence a Polish space and a Borel injection \( \iota: X/E \to Y \) that admits a Borel lifting (i.e., a Borel map \( f: X \to Y \) such that \( f(x) = \iota([x]_E) \)). Clearly, for any standard Borel space \( X \), the equality relation \( =_X \) is smooth.

Smooth equivalence relations admit elements of some Polish space as complete invariants. Here are some natural examples.

**Example 2.2.** (1) Isomorphism on countable divisible abelian groups. The isomorphism relation on the standard Borel space of countable divisible abelian groups is smooth. To see this, recall that if \( A \) is a countable divisible abelian group, then \( A = D \oplus T \), where \( T \) is the torsion subgroup and \( D \) is torsion-free. Let \( r_0(A) \in \mathbb{N} \cup \{\infty\} \) be the rank of \( D \); and for each prime \( p \), let \( r_p(A) \in \mathbb{N} \cup \{\infty\} \) be the rank of the \( p \)-component \( T_p \) of \( T \). Then the invariant

\[
\rho(A) = (r_0(A), r_2(A), r_3(A), \ldots, r_p(A), \ldots) \in (\mathbb{N} \cup \{\infty\})^\mathbb{N}
\]

determines \( A \) up to isomorphism.

(2) Isomorphism of Bernoulli shifts. Let \((X, \mu)\) be a probability space (\( X \) can be finite) and let \( \mu^\mathbb{Z} \) denote the product measure on \( X^\mathbb{Z} \). Let \( S: X^\mathbb{Z} \to X^\mathbb{Z} \) denote the shift automorphism; i.e., for \( f \in X^\mathbb{Z} \) and \( n \in \mathbb{N} \), \( S(f)(n) = f(n-1) \). The dynamical system \((X^\mathbb{Z}, \mu^\mathbb{Z}, S)\) is called a Bernoulli shift. By the measure isomorphism theorem, every Bernoulli shift is isomorphic to \(([0,1], \lambda, T)\), where \( \lambda \) is the Lebesgue measure and \( T \) some measure-preserving automorphism of \(([0,1], \lambda)\). In this case, we would call \( T \) a Bernoulli shift as well, and let \( B \subseteq \text{Aut}([0,1], \lambda) \) be the set of all Bernoulli shifts. (Here \( \text{Aut}([0,1], \lambda) \) is the Polish space space of all measure-preserving automorphisms of \(([0,1], \lambda)\) up to a.e. equality equipped with the weak topology.) Ornstein showed that \( B \) is a Borel subset of \( \text{Aut}([0,1], \lambda) \), and hence is a standard Borel space. Furthermore, one can assign to each \( T \in B \) a real number \( e(T) \) called entropy. While the notion of entropy was introduced by Kolmogorov, Ornstein [36] also showed that \( e(T) \) completely classify \( T \) up to isomorphism: that is \( T_1 \cong T_2 \iff e(T_1) = e(T_2) \).

Now we see some useful characterizations of smoothness. A **transversal** for \( E \) is a set \( B \subseteq X \) which intersect each \( E \)-class in exactly one point. A **selector** for \( E \) is a function \( s: X \to X \) such that \( s(x) \in E \) and \( x \in E \) implies \( s(x) = s(y) \).

**Theorem 2.3.** Let \( E \) be a countable Borel equivalence relation on \( X \). The following are equivalent:

(i) \( E \) is smooth;
(ii) \( E \) admits a Borel transversal;

(iii) \( E \) admits a Borel selector.

Proof. \( (i) \implies (ii) \) Let \( Y \) be a Polish space and \( f: X \to Y \) witness smoothness for \( E \). Clearly \( f \) is a countable-to-one function, and its graph is a Borel set of \( X \times Y \) with countable sections. It follows by Luzin-Novikov theorem that \( \text{proj}_Y(\text{Graph}(f)) = f(X) \) is Borel (cf. Exercise 1.16), moreover there is a Borel uniformizing function \( g: f(X) \to X \). Then, the set \( g \circ f(X) \) is a Borel transversal for \( E \).

(ii) \( \implies (iii) \) Let \( A \subseteq X \) be a Borel transversal for \( E \). Define \( s: X \to X \) by setting \( s(x) = a \in A \) such that \( a E x \). It is clear that \( s \) is a selector for \( E \) and \( \text{Graph}(s) = \{(x, y) \in E : y \in A\} \) is a Borel set, thus \( s \) is Borel by Theorem 1.11.

(iii) \( \implies (i) \) A Borel selector \( s: X \to X \) witnesses smoothness as \( x E y \) if and only if \( s(x) = s(y) \).

\( \square \)

Remark 2.4. Note that the implication \( (iii) \implies (ii) \) is immediate because if \( s: X \to X \) is a Borel selector for \( E \), then we can define a Borel transversal by \( \{x : s(x) = x\} \).

On the other hand, the equivalence between \( (i) \) and \( (ii)-(iii) \) does not hold if we drop countability. In fact, there exists a Borel \( X \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \) without Borel uniformization and such that \( \text{proj}_1 X = \mathbb{N}^\mathbb{N} \) and \( X \) (e.g., see [22, Exercise 18.17]). Next, we define \( E \) on \( X \) by declaring \( x E y \iff \text{proj}_1(x) = \text{proj}_1(y) \). Obviously, the map \( \text{proj}_1: X \to \mathbb{N}^\mathbb{N} \) witnesses smoothness of \( E \), but \( E \) does not admit a Borel selector, whose existence would be equivalent to Borel uniformization for \( X \).

Exercise 2.5. Show that the action \( \mathbb{Z} \rtimes \mathbb{R} \) by \( n \cdot x = x + n \) induces a smooth equivalence relation.

Exercise 2.6. Show that every finite Borel equivalence relation is smooth.

A countable Borel equivalence relation \( E \) is called \textit{aperiodic} if every \( E \)-class is infinite.

Proposition 2.7. Suppose that \( E \) is an aperiodic countable smooth Borel equivalence relation on a standard Borel space \( X \). Then there is a sequence \( (B_n)_n \) of pairwise disjoint Borel transversals of \( E \) such that \( X = \bigcup_n B_n \).

Proof. Let \( Y \) be a standard Borel space and \( f: X \to Y \) be a Borel reduction from \( E \) to \( =_Y \). The set \( \text{Graph}(f) \) is Borel whose horizontal sections are countable. Then \( \text{Graph}(f) = \bigcup_n g_n \) for injective
Borel partial functions $g_n : f(X) \to X$. Then $A_n = \text{ran } g_n$ is a Borel partial transversal for $\bigcup_n A_n = X$.

Define $k_n : X \to \mathbb{N}$ by setting $k_n(x) = \min \{ k \in \mathbb{N} : A_k \cap [x]_E \nsubseteq \bigcup_{m < n} A_{k_m(x)} \}$. Then let $B_n = \bigcup_{k \in \mathbb{N}} \{ x \in A_k : k = k_n(x) \}$. These sets are pairwise disjoint transversal for $E$. Each $B_n$ is Borel (why?) and $X = \bigcup_{i \in \mathbb{N}} B_n$.

A (countable) separating family for $E$ is a sequence $\{ A_n \}_{n \in \mathbb{N}}$ of subsets of $X$ such that

$x E y \iff \forall n (x \in A_n \iff y \in A_n)$.

**Proposition 2.8.** A countable Borel equivalence relation is smooth if it admits a countable separating family consisting of Borel sets.

**Proof.** Exercise.

We let $X/E$ denote the equivalence classes with the quotient Borel structure, i.e. $B \subseteq X/E$ is said to be Borel if and only if its inverse image in $X$ is Borel.

**Proposition 2.9.** A countable Borel equivalence relation $E$ is smooth if and only if the quotient $X/E$ is a standard Borel space.

**Proof.** First assume that $Z$ is a Polish space and $f : X \to Z$ witnesses smoothness. By countability of $E$, it follows that $f$ is countable-to-one, so $Y = f(X)$ is a Borel set by Exercise 1.16. Then $Y$ is a standard Borel space with the induced Borel structure. Using the fact that $f$ is countable-to-1 we can check that the bijection $X/E \to Y$ induced by $f$ is a Borel isomorphism. It follows that $X/E$ is standard.

Conversely, if the induced Borel structure on $X/E$ is standard, then the map $x \mapsto [x]_E$ witnesses that $E$ is smooth.

**Ergodicity**

In this section we introduce ergodicity, a measure theoretical obstruction to smoothness. Let $(X, \mathcal{B})$ be a Borel space. A (Borel) measure on $X$ is a function $\mu : \mathcal{B} \to \mathbb{R}^+ \cup \{ \infty \}$ that takes $\emptyset$ to 0 and that is $\sigma$-additive, i.e. for pairwise disjoint Borel sets $A_n$, for $n \in \mathbb{N}$, we have

$$\mu \left( \bigcup_n A_n \right) = \sum_n \mu(A_n).$$
If \( \mu \) is a measure on \( X \) and \( Y \subseteq X \) is a Borel set, then \( \mu \upharpoonright Y \) is the measure on \( Y \) which is the restriction of \( \mu \) to the Borel subsets of \( Y \). Moreover, when \( f : X \to Y \) is a Borel function between Polish spaces and \( \mu \) is a Borel measure on \( X \), we can define the push-forward measure \( f_*\mu \) on \( Y \) by setting \( f_*\mu(B) = \mu(f^{-1}(B)) \), for every Borel \( B \subseteq Y \).

We call a Borel measure \( \mu \) on \( X \) finite if \( \mu(X) < \infty \). In case \( \mu(X) = 1 \), we call \( \mu \) a probability measure. Until further notice we assume that \( \mu \) is a finite Borel measure. A set \( A \subseteq X \) is called \( \mu \)-measurable if there is some Borel \( B \subseteq X \) so that \( \mu(A \Delta B) = 0 \). In this case, we set \( \mu(A) = \mu(B) \).

**Definition 2.10.** A measure \( \mu \) on \( X \) is said to be \( E \)-ergodic if for every measurable \( A \subseteq X \), if \( A \) is \( E \)-invariant then \( \mu(A) = 0 \) or \( \mu(X \setminus A) = 0 \). A measure \( \mu \) is called non-atomic if every singleton has measure 0.

**Proposition 2.11.** Let \( E \) be a countable Borel equivalence relation on a standard Borel space. If \( X \) has an \( E \)-ergodic, nonatomic measure \( \mu \), then \( E \) is not smooth.

**Proof.** Suppose that \( E \) is smooth towards contradiction. Let \( \{A_n\}_{n \in \mathbb{N}} \) be a countable Borel generating family for \( E \). Each \( A_n \) is invariant, so either \( \mu(A_n) = 0 \) or \( \mu(X \setminus A_n) = 0 \) by ergodicity. Then the set

\[
C = \bigcap\{A_n : \mu(X \setminus A_n) = 0\} \cap \bigcap\{X \setminus A_n : \mu(A_n) = 0\}
\]

has positive measure. However, \( C = [x]_E \) for some \( x \in X \). It follows that \( \mu(C) = 0 \), since \( \mu \) is non-atomic.

**Definition 2.12.** For \( x, y \in 2^\mathbb{N} \) we define \( x E_0 y \iff \exists m \forall n \geq m \ (x(n) = y(n)). \)

**Fact 2.13.** The product probability measure on \( 2^\mathbb{N} \) (where each bit in \( \{0, 1\} \) has measure \( \frac{1}{2} \)) is non-atomic and \( E_0 \)-ergodic. Thus \( E_0 \) is not smooth.

**Generic ergodicity**

In this subsection we discuss another restriction to smoothness. Until further notice, let \( X \) be a Polish space.

**Definition 2.14.** A set \( A \subseteq X \) is nowhere dense if its closure has nonempty interior.

Examples of nowhere dense sets include \( \mathbb{Z} \) and \( \{\frac{1}{n} : n \in \mathbb{N}^+\} \) in \( \mathbb{R} \).

**Definition 2.15.** \( A \subseteq X \) is meager if it is countable union of nowhere dense sets. We call \( A \subseteq X \) comeager if its complement is meager.
An example of meager set is $\mathbb{Q}$ in $\mathbb{R}$. Likewise the sets of measure 0 with respect to a given measure, the set of all meager subsets of a topological space forms a $\sigma$-ideal. Hence we can define a notion of measurability with respect to meager sets.

**Definition 2.16.** $A \subseteq X$ is Baire measurable if there is an open set $U \subseteq X$ such that $A \Delta U$ is meager. Moreover, we call a function $f : X \to Y$ Baire measurable if the preimage of every open set in $Y$ is Baire measurable.

**Fact 2.17.** The set of Baire measurable subsets of $X$ is the smallest $\sigma$-algebra containing the open sets and the meager sets.

In particular, all Borel sets are Baire measurable. The existence of subsets of $\mathbb{R}$ that are not Baire measurable is independent from ZF. Using the axiom of choice it is possible to build a so-called Bernstein set that is not Baire measurable. On the other hand, the Axiom of Determinacy (AD) implies that all subsets of a Polish space are Baire measurable.

**Definition 2.18.** An equivalence relation on $X$ is called generically ergodic if every $E$-invariant Baire measurable set is either meager or comeager.

**Proposition 2.19.** Suppose that $E$ is an equivalence relation on $X$ and $f : X \to 2^\mathbb{N}$ is a Baire measurable homomorphism between $E$ and $\Delta(2^\mathbb{N})$ (i.e., $x E y$ implies $f(x) = f(y)$). If $E$ is generically ergodic, then there is $x_0 \in 2^\mathbb{N}$ such that $f^{-1}(x_0)$ is comeager.

**Proof.** We define an increasing sequence $(s_n)_n$ of finite binary sequences such that $|s_n| = n$ and $f^{-1}(N_{s_n})$ is comeager. Put $s_0 = \emptyset$. As inductive hypothesis we assume that $f^{-1}(N_{s_n})$ is comeager. Notice that since $f$ is a Baire measurable homomorphism, the preimage of every open set is $E$-invariant and Baire measurable. So, $f^{-1}(N_t)$ is either meager or comeager by generic ergodicity. Since we have

$$f^{-1}(N_{s_n}) = f^{-1}(N_{s_n} \sim 0) \cup f^{-1}(N_{s_n} \sim 1),$$

there is some $i \in \{0, 1\}$ such that $f^{-1}(N_{s_n} \sim i)$ is comeager. Set $s_{n+1} = s_n \sim i$ and put $x_0 = \bigcup_n s_n$. It follows that

$$f^{-1}(x_0) = f^{-1}(\bigcap_{n} N_{s_n}) = \bigcap_{n} f^{-1}(N_{s_n})$$

which is a countable intersection of comeager sets, hence comeager.

**Corollary 2.20.** Let $E$ be an equivalence relation on $X$. If $E$ is generically ergodic and every equivalence class is meager, then $E$ is not smooth.
Proof. Suppose that \( f: X \to 2^\mathbb{N} \) is a Borel reduction. In particular, \( f \) is a Baire measurable reduction, hence the preimage of every point is a single \( E \)-class, which is meager by the assumption. However, by Proposition 2.19, there is a point in \( 2^\mathbb{N} \) whose preimage is comeager. \( \square \)

**Fact 2.21.** Suppose \( G \curvearrowright X \) in a continuous fashion. The following are equivalent:

1. \( E_G \) is generically ergodic.
2. Every invariant nonempty open set is dense.
3. For comeager-many \( x \in X \), the orbit \( G \cdot x = [x] \) is dense.
4. There is a dense orbit.
5. For every nonempty open sets \( U, V \subseteq X \), there is \( g \in G \) such that \( (gU) \cap V \neq \emptyset \).

**Example 2.22.** The Vitali equivalence relation is generated by the action \( Q \curvearrowright \mathbb{R} \) by \( q \cdot x = x + q \).

**Exercise 2.23.** Show that \( E_0 \) is not smooth using generic ergodicity.

## 3 Structural results

Suppose that \( E \) and \( F \) are Borel equivalence relations on standard Borel spaces \( X \) and \( Y \), respectively.

**Definition 3.1.** We say that \( E \) is **Borel reducible** to \( F \) (in symbols \( E \leq_B F \)) if and only if there exists a Borel function \( f: X \to Y \) such that \( x E y \iff f(x) F f(y) \). Such function \( f \) is called **Borel reduction**.

If moreover \( f \) is injective we say that \( F \) is a **Borel embedding** and \( E \) is **Borel embeddable** into \( F \) (written \( E \subseteq_B F \)).

**Exercise 3.2.** Suppose that \( E \) is a Borel equivalence relation on a standard Borel space \( X \) with infinitely many equivalence classes. Let \( x_0 \in X \). Then \( E \) is Borel reducible to \( E \upharpoonright (X \setminus \{x_0\}E) \).

Occasionally we will be using a weaker notion to compare equivalence relations.

**Definition 3.3.** A function \( h: X \to Y \) is a **Borel homomorphism** between \( E \) and \( F \) if it is Borel and \( x E y \implies f(x) F f(y) \).

**Theorem 3.4** (Silver’s dichotomy [38]). For every Borel (in fact coanalytic) equivalence relation on a standard Borel space, exactly one of the following holds:
(I) There are only countably many E-classes;

(II) $\text{id}_\mathbb{R} \subseteq_B E$.

The original prove of Silver original is quite involved and uses a forcing argument. Another proof was recently found by Miller who deduced Silver’s dichotomy from a dichotomy result of Kechris, Solecki, and Todorcević Borel combinatorics on Borel colorings for analytic graphs [25]. Another alternative proof of Silver’s dichotomy was given by Harrington, who used the so-called Gundy-Harrington topology and methods from recursion theory. Interestingly, the Gundy-Harrington topology and those effective methods are key in the proof of Harrington, Kechris, and Louveau of another remarkable dichotomy theorem on Borel equivalence relations.

**Theorem 3.5 (Glimm-Effros dichotomy [15]).** Let $E$ be a Borel equivalence relation on a standard Borel space. Then either

(I) $E$ is smooth; or

(II) $E_0 \subseteq_B E$.

The reason why Theorem 3.5 is named after Glimm and Effros is because it generalizes results of both from the ’60s. In fact, Glimm [14] proved a particular case of Theorem 3.5 for equivalence relations induced by continuous action of locally compact Polish groups. Glimm’s result was generalized afterwards by Effros [8] to $F_\sigma$ orbit equivalence relation induced by continuous actions of Polish groups.

Combining Theorem 3.4 with Theorem 3.5 it is clear that an initial segment of the class of countable Borel equivalence relation have the following linear structure

$$\text{id}(1) <_B \text{id}(2) <_B \text{id}(3) <_B \cdots <_B \text{id}(\mathbb{N}) <_B \text{id}(\mathbb{R}) <_B E_0 \leq_B \cdots$$

**Universality**

At the other extreme of smoothness we encounter the phenomenon universality.

**Definition 3.6.** A countable Borel equivalence relation $F$ is called *universal* if $E \leq_B F$ for every countable Borel equivalence relation $E$.  

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Given any standard Borel space \( X \) and a countable group \( G \), denote by \( X^G \) the set of functions from \( G \) to \( X \) with the product Borel structure. The shift action \( G \) on \( X^G \) given by \( g \cdot f(a) = f(g^{-1}a) \) for \( f \in X^G, g \in G \).

**Definition 3.7.** We denote by \( E(G, X) \) the orbit equivalence relation generated by the shift action of \( G \) on \( X^G \).

**Exercise 3.8.** If \( G \) is homomorphic image of \( H \), then \( E(G, X) \leq_B E(H, X) \).

**Exercise 3.9.** If \( G \) is a subgroup of \( H \), then \( E(G, X) \leq_B E(H, X) \).

Let \( \mathbb{F}_\infty \) be the countable non-abelian free group on infinitely many generators.

**Theorem 3.10.** \( E_\infty = E(\mathbb{F}_\infty, (2^N)^{F_2}) \) is a universal countable Borel equivalence relation.

**Proof.** Let \( E = E^X_G \) for some countable \( G \rhd X \). Since any countable group is homomorphic image of \( \mathbb{F}_\infty \), we can suppose that \( E \) is induced by some Borel action \( \mathbb{F}_\infty \rhd X \).

Since \( X \) is standard Borel there exists a sequence \( \{U_n\}_{n \in \mathbb{N}} \) of Borel subsets of \( X \) which separates points. Define a map \( f: X \rightarrow (2^N)^{F_2} \) by

\[
 f_x(a)(i) = 1 \iff a^{-1} \cdot x \in U_i
\]

It is immediate that \( f \) is injective, moreover \( x \, E \, y \iff f_x \, E_\infty \, f_y \) because

\[
 g \cdot f_x(a)(i) = 1 \iff f_x(g^{-1}a)(i) = 1
\]

\[
 \iff (a^{-1}g)x \in U_i
\]

\[
 \iff a^{-1}(gx) \in U_i
\]

\[
 \iff f_{gx}(a)(i) = 1.
\]

The following Borel reductions were obtained by Dougherty, Jackson, and Kechris [6].

**Fact 3.11.** \( E(G, 2^{\mathbb{Z} - \{0\}}) \leq_B E(G \times \mathbb{Z}, 3) \)

**Fact 3.12.** \( E(G, 3) \leq_B F(G \times \mathbb{Z}_2, 2) \).

**Proposition 3.13.** \( E(\mathbb{F}_2, 2) \) is a universal countable Borel equivalence relation.

**Proof.** Since \( \mathbb{F}_2 = \langle a, b \rangle \) has a subgroup isomorphic to \( \mathbb{F}_\infty \) (for example take \( \langle a^n b^n a^{-n} \mid n \geq 1 \rangle \)), it suffices to show that \( E(\mathbb{F}_\infty, 2) \) is universal. Then we have

\[
 E(\mathbb{F}_\infty, 2^N) \leq_B E(\mathbb{F}_\infty, 2^{\mathbb{Z} - \{0\}}) \leq_B E(\mathbb{F}_\infty \times \mathbb{Z}, 3) \leq_B E(\mathbb{F}_\infty \times \mathbb{Z} \times \mathbb{Z}_2, 2) \leq_B E(\mathbb{F}_\infty, 2) \]

\( \square \)

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It follows that every group containing a nonabelian free subgroup induces a universal countable Borel equivalence relation. Whether the converse is also true is still open.

**Definition 3.14.** A countable group $G$ is said to be action universal if there is some standard Borel space $X$ and some Borel action $G \acts X$ so that $E_G^X$.

**Problem 3.15.** Is it true that if $G$ is action universal then $G$ contains a nonabelian free subgroup?

Recent work of Thomas [44] suggests that the answer might be negative.

### 4 Hyperfiniteness

**Definition 4.1.** A Borel equivalence relation $E$ is hyperfinite if there is an increasing sequence of finite Borel equivalence relations $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$ such that $E = \bigcup_n F_n$.

Notice that all hyperfinite equivalence relations are in fact countable.

**Example 4.2.** (1) $E_0$ is hyperfinite. To see this define $F_n$ by

$$x F_n y \iff \forall m \geq n \ (x(m) = y(m)).$$

It is clear that $E_0 = \bigcup_n F_n$.

(2) If $G$ is a Polish group (i.e., a topological group whose topology is Polish) and $H$ is a subgroup of $G$ that can be written as $H = \bigcup H_n$ such each $H_n$ is a countable discrete subgroup of $G$ and $H_n \subseteq H_{n+1}$ for $n \geq 1$. Then the coset equivalence relation on $G$, which is defined by

$$g E_H^G h \iff h \in gH,$$

is hyperfinite. As an application we obtain that $E_v$ is hyperfinite as $Q = \bigcup_n \frac{1}{n} \mathbb{Z}$.

**Proposition 4.3.** If $E$ smooth, then $E$ is hyperfinite.

**Proof.** Let $E = E_G^X$ for some countable group $G$. Suppose that $G = \{g_0, g_1, \ldots\}$ with $g_0 = 1$. Since $E$ is smooth, $E$ admits a Borel selector $s \colon X \to X$ by Proposition 2.3–(iii). Then define an sequence of finite Borel equivalence relations $F_n$, for $n \geq 1$, by

$$x F_n y \iff \left( \bigvee_{i=0}^{n} x = g_i s(x) \text{ and } \bigvee_{i=0}^{n} y = g_i s(x) \right) \text{ or } x = y.$$ 

Then for any $F_n$, each class is either a singleton or has at most $n + 1$ elements. Hence $F_n$ is a finite Borel equivalence relation and $E = \bigcup_n F_n$. 

\[\square\]
The following closure properties are easy to verify.

**Fact 4.4.**

(i) If $E \subseteq F$ and $F$ is hyperfinite, then $E$ is hyperfinite.

(ii) If $E$ is hyperfinite and $Y \subseteq X$ is Borel, then $E \upharpoonright Y$ is hyperfinite.

Given an equivalence relation $E$ on $X$, a subset $A \subseteq X$ is a complete section for $E$ if it meets every $E$-class (i.e., for all $x \in X$, $[x]_E \cap A \neq \emptyset$).

**Proposition 4.5.** Let $E$ be a countable Borel equivalence relation on $X$. If $A \subseteq X$ is a Borel complete section for $E$ such that $E \upharpoonright A$ is hyperfinite, then $E$ is hyperfinite.

**Proof.** Suppose that $F_0 \subseteq F_1 \subseteq \cdots$ are finite Borel equivalence relations on $A$ such that $E \upharpoonright A = \bigcup_n F_n$. Let $E = E_G$ and $G = \{g_0, g_1, \ldots\}$ such that $g_0 = 1$. For each $x \in X$ let $n(x)$ be the least index such that $g_{n(x)} x \in A$. Define $\tilde{F}_n$ by

$$x \tilde{F}_n y \iff (n(x) \leq n(y) \leq n(y) \text{ and } g_{n(x)} x F_n g_{n(y)} y \text{ or } x = y).$$

Each $\tilde{F}_n$ is a finite Borel equivalence relation. So $E$ is hyperfinite as $E = \bigcup_n \tilde{F}_n$. □

**Exercise 4.6.** If $F$ is hyperfinite and $E \leq_B F$, then $E$ is hyperfinite.

**Proposition 4.7 ([6]).** Let $E$ be a countable Borel equivalence relation. Then the following are equivalent:

(i) $E$ is hyperfinite.

(ii) $E$ is hypersmooth, i.e., $E = \bigcup_{n \in \mathbb{N}} F_n$ where $(F_n)_{n \in \mathbb{N}}$ is an increasing sequence of smooth Borel equivalence relations.

(iii) There is a Borel action $a: \mathbb{Z} \curvearrowright X$ such that $E = E_a$

**Proof.** The implication (1) $\implies$ (2) is clear since every finite Borel equivalence relation is smooth (see 2.6). For (2) $\implies$ (1) we refer the reader to [6].

For (1) $\implies$ (3), first suppose that $E$ is smooth. Notice that $X$ can be partitioned into $\bigcup_{n \in \mathbb{N} \cup \{\infty\}} X_n$ with $X_n = \{x \in X : \# [x]_E = n\}$. Clearly each $X_n$ is $E$-invariant and $E \upharpoonright X_n$ is smooth. We may assume that each $E$-class has the same cardinality. If each $E$-class has cardinality $n \in \mathbb{N}$, let $<$ be a Borel well-ordering of $X$. Then, define

$$T(x) = \begin{cases} x^+ & \text{if } x \text{ has an immediate successor in } < \upharpoonright [x]_E \\ \min [x]_E & \text{otherwise.} \end{cases}$$

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Clearly $T$ generates a Borel $\mathbb{Z}$-action by $n \cdot x = T^n(x)$ and this action induces $E$. On the other hand, if $E$ is aperiodic, then there exists a sequence $(B_n)_{n \in \mathbb{Z}}$ of pairwise disjoint of Borel transversals for $E$ so that $X = \bigcup_n B_n$ (see Proposition 2.7.) For any $x \in B_n$, we define $T(x) = y$ for the unique $y \in X$ such that $x E y$ and $y \in B_n$. It is clear that $E$ is induced by the $\mathbb{Z}$-action generated by $T$.

Now assume that $E$ is hyperfinite. Let $(F_n)_{n \in \mathbb{N}}$ be an increasing sequence $(F_n)n \in \mathbb{N}$ of finite Borel equivalence relations on $X$ such that $F_0$ is equality and $E = \bigcup_n F_n$. Also let $<$ be a Borel linear ordering on $X$.

Since each $F_n$ is finite equivalence relation, it is smooth. So there exists a Borel selector $s_n: X \to X$ for each $F_n$. Given distinct $x, y \in X$, let $n(x, y)$ denote the maximal natural number $n$ for which $s_n(x) \neq s_n(y)$, and put $x \lessdot y \iff s_n(x) < s_n(y)$. Then $<$ is a Borel partial order on $X$ whose restriction to any $E$-class is isomorphic to the usual ordering of $\mathbb{N}$, $-\mathbb{N}$, or $\mathbb{Z}$. Notice that $E$ is smooth on the union of the $E$-classes $C$ for which $<|C$ is not isomorphic to $\mathbb{Z}$ because we can use the existence of a maximal or a minimal element to define a Borel transversal. On the remaining classes we can define a $\mathbb{Z}$-action generating $E$ in the natural way.

The proof of $(3) \Rightarrow (1)$ relies on the so called Marker lemma.

**Lemma 4.8.** Every aperiodic countable Borel equivalence relation admits a vanishing sequence of markers, i.e., a sequence of Borel complete sections that intersect every $E$-class.

In each $E$-class $C$ we define the $\mathbb{Z}$-ordering

$$x \lessdot_C y \iff \exists n > 0 \ (n \cdot x = y).$$

Let $(A_n)_n$ be a vanishing sequence of markers for $E$. Note that the union of the set of all classes $C$ in which $A_n \cap C$ has a least or largest element in $<_{C}$, for some $n \in \mathbb{N}$, is an invariant Borel set on which $E$ is smooth. So we can assume that for each class $C$, $A_n \cap C$ is unbounded in both directions in $<_{C}$.

Then define $F_n$, for $n \in \mathbb{N}$, by

$$x F_n y \iff x = y \text{ or } (x E y \text{ and } [x,y] \cap B_n = \emptyset),$$

where $[x,y] = \{z \in [x]_E \mid x < z < y\}$. We have $E = \bigcup_n F_n$. \hfill \Box

In view of 3.5 $E_0$ is minimal among the nonsmooth hyperfinite Borel equivalence relations. It turns out that $E_0$ is also a universal element for this class.

**Theorem 4.9 ([6, Theorem 7.1]).** If $E$ is Borel hyperfinite, then $E \leq B E_0$. 

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As a consequence, being reducible to $E_0$ is equivalent to (1)–(3) of Theorem 4.7. An alternative proof of Theorem 4.9 was given by Hjorth (see [46]).

It follows from Theorem 4.9 that all nonsmooth hyperfinite equivalence relation are Borel bi-reducible. However, hyperfinite Borel equivalence relation admits have a finer classification up to Borel isomorphism.

**Definition 4.10.** Let $E$ be a countable Borel equivalence relation on a standard Borel space $X$. A (probability) measure on $X$ is said $E$-invariant if there is some countable $G$ acting on $X$ in a Borel fashion and $E$ is invariant under this action. Denote by $\text{EINV}(E)$ the set of $E$-ergodic $E$-invariant probability measures on $X$.

For any countable Borel equivalence relation $E$, $|\text{EINV}(E)| \in \{0, 1, 2, \ldots, n, \ldots, \aleph_0, 2^{\aleph_0}\}$. By results of Farrel [9] and Varadarajan [47] a countable Borel equivalence relation $E$ that admits an $E$-invariant probability measure, also admits an $E$-ergodic one.

**Theorem 4.11 ([6]).** Let $E, F$ be aperiodic, non-smooth hyperfinite Borel equivalence relations. Then $E \cong_B F$ if and only if $|\text{EINV}(E)| = |\text{EINV}(F)|$.

For example $E_0$ has a unique invariant, ergodic, probability measure, while the orbit relation $F(\mathbb{Z}, 2)$ induced by the shift action of $\mathbb{Z}$ on the free part of $2^\mathbb{Z}$ admits $2^{\aleph_0}$ invariant ergodic probability measures (consider product measures corresponding to the $(p, 1 - p)$ measure on $\{0, 1\}$ for $0 < p < 1$).

**The union problem**

We pointed out that hyperfiniteness coincide with hypersmoothness. It is natural to ask whether hyper-hyperfinite implies hyperfinite. This is an interesting major open problem in this area.

**Question 4.12 (Union problem).** Let $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ be an increasing sequence of hyperfinite equivalence relations. Is $E = \bigcup_n E_n$ hyperfinite?

**Theorem 4.13 (Dye [7]; Krieger [26]).** Let $X$ be a standard Borel space and $\mu$ a probability measure on $X$. If $E$ is the union of an increasing sequence of hyperfinite Borel equivalence relations, then $E$ is hyperfinite $\mu$-a.e., i.e., there is a conull Borel set $A \subseteq X$ such that $E \upharpoonright A$ is hyperfinite.

Unfortunately to attack this and other problem about hyperfiniteness we cannot use Baire category methods. In fact, every countable Borel equivalence relation is hyperfinite generically.
**Theorem 4.14** (Hjorth-Kechris). Suppose that $E$ is a countable Borel equivalence relation on a Polish space $X$. Then there is an $E$-invariant comeager Borel set $C \subseteq X$ such that $E \upharpoonright C$ is hyperfinite.

This is far from true in measure theoretical context. For example, consider the equivalence relation $E(F_2, 2)$ on $X = 2^{F_2}$ with the usual product measure. For every conull Borel set $B \subseteq X$, the restriction $E \upharpoonright A$ is not hyperfinite.

**Amenable groups**

Suppose that $X$ is a set. A finitely additive probability measure (f.a.p.m.) on $A$ is a map

$$
\mu : \mathcal{P}(X) \to [0, 1]
$$

such that $\mu(X) = 1$ and whenever $A, B \subseteq X$ are disjoint then $\mu(A \cup B) = \mu(A) + \mu(B)$.

If $G$ is a countable group and $\mu$ is a f.a.p.m. on $G$ we say that $\mu$ is left-invariant if $\mu(gA) = \mu(A)$ for all $g \in G$ and $A \subseteq G$.

**Definition 4.15.** A countable group is amenable if it admits a left-invariant finitely additive probability measure.

**Example 4.16.**

1. Every finite group $G$ is amenable with left-invariant f.a.p.m. $\mu(A) = |A|/|G|$. 
2. $\mathbb{Z}$ is amenable. In fact, if $\mathcal{U}$ is a nonatomic ultrafilter on $\mathbb{N}$ we can define a left-invariant f.a.p.m. by $\mu(A) = \lim_{n \in \mathcal{U}} \frac{|A \cap \{-n, \ldots, n\}|}{2n+1}$.

**Fact 4.17.**

(i) If $H$ is amenable and $G$ is a subgroup of $H$, then $G$ is amenable.

(ii) If $N$ is a normal subgroup of $G$, $G$ is amenable if and only if both $N$ and $G/N$ are amenable. In particular, amenable groups are closed under epimorphic images.

(iii) $G$ is amenable if and only if every finitely generated subgroup is amenable.

If $X, Y \subseteq X$ we write $X \sim Y$ if there are partitions $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ of $X, Y$ respectively and group elements $g_1, \ldots, g_n \in G$ such that $g_iX_i = Y_i$.

**Definition 4.18.** A group $G$ is paradoxical if there are disjoint sets $A, B \subseteq G$ such that $A \sim B \sim G$.

**Proposition 4.19.** If $G$ is paradoxical, then $G$ is not amenable.
Proof. Suppose that $\mu$ is a left-invariant finitely additive probability measure on $G$, and fix disjoint $A, B \subseteq G$ such that $A \cap B = \emptyset$ and $A \sim B \sim G$.

$$\mu(G) \geq \mu(A \cup B) = \mu(A) + \mu(B) = 2\mu(G)$$

which is a contradiction. \hfill \qed

In fact, we have the following:

**Theorem 4.20.** $G$ is not amenable if and only if $G$ is paradoxical.

**Proposition 4.21.** The nonabelian free group on two generators $\mathbb{F}_2$ is paradoxical.

**Proof.** Let $\mathbb{F}_2 = \langle a, b \rangle$. For each $x \in \{a^\pm 1, b^\pm 1\}$, let $S(x)$ be the set of reduced words which begin with $x$.

Let $A = S(a) \cup S(a^{-1})$ and $B = S(b) \cup S(a^{-1})$. Since $aS(a^{-1}) = \mathbb{F}_2 \setminus S(a)$, it follows that $A \sim \mathbb{F}_2$. Similarly, we get $B \sim \mathbb{F}_2$, thus $\mathbb{F}_2$ is paradoxical. \hfill \qed

Another characterization of amenability uses measure theory.

Let $X$ be a separable and metrizable topological space. Denote by $P(X)$ the set of all probability measure on $X$. We give $P(X)$ the topology whose basis is formed by the sets

$$U_{\mu,\epsilon,f_1,\ldots,f_n} = \{\mu \in P(X) : |\int f_i d\nu - \int f_i d\mu| < \epsilon\}$$

for any $i = 1, \ldots, n$, $\mu \in P(X)$, $\epsilon > 0$, $f_i$ bounded real valued function on $X$.

**Fact 4.22 ([22, Section 17.E]).** If $X$ is Polish, then $P(X)$ is Polish. If moreover $X$ is compact, then $P(X)$ is a compact Polish space.

Note any continuous action $G \curvearrowright X$ induces a continuous $G$-action on $P(X)$ by

$$g \mu(A) = \mu(g^{-1}A),$$

for all $\mu$ measurable $A \subseteq X$.

**Definition 4.23.** We say that $\mu \in P(X)$ is $G$-invariant if $g \mu = \mu$ for all $g \in G$.

**Theorem 4.24.** Let $G$ be a countable group. Then $G$ is amenable if and only if for every compact metric space $K$ and every continuous action $G \curvearrowright K$ there exists a $G$-invariant measure in $P(K)$.

With this in mind we can prove the following:
Theorem 4.25. Suppose that G acts on X freely, and there is a G-invariant probability measure µ on X. If \( E^X_G \) is hyperfinite, then G is amenable.

Suppose that G acts on X, and H acts on Y freely. If \( f \) is a Borel homomorphism from \( E^X_G \) to \( E^Y_H \) then we can define a Borel map \( \alpha: G \times X \to H \) by

\[
\alpha(g, x) = h \iff h \cdot f(x) = f(g \cdot x).
\]

Notice that, since the action of H on X is free, \( \alpha \) is well-defined. Moreover it is straightforward that \( \alpha \) satisfies

\[
\alpha(gh, x) = \alpha(g, hx)\alpha(h, x)
\]

for all \( g, h \in G \) and \( x \in X \). A map with such property is called Borel cocycle.

Definition 4.26. Suppose that \( \alpha: G \times X \to H \) is a Borel cocycle and H acts on Y in a Borel way. A \( \mu \)-measurable map \( f: X \to Y \) is called \( \alpha \)-invariant if

\[
\alpha(g, x)f(x) = f(gx)
\]

\( \mu \)-almost everywhere.

Theorem 4.27 (Zimmer; see also [19, B 3.1]). Let G be a countable amenable group acting on a standard Borel space X with invariant probability measure \( \mu \) in a Borel fashion. Moreover let H be a countable group acting continuously on a compact metric space K. For any cocycle \( \alpha: G \times X \to H \), there exists an \( \alpha \)-invariant \( \mu \)-measurable map from \( X \to P(K) \), \( x \mapsto \nu_x \).

Proof of Theorem 4.25. Let G act freely on X, and \( \mu \) be a G-invariant probability measure. Since \( E \) is hyperfinite \( E = E_X \) for some Borel action of \( Z \) on X. The free action of G on X induces a cocycle \( \alpha: Z \times X \to H \) by \( \alpha(n, x) = g \) if and only if \( nx = gx \).

Now suppose that K is a compact metric space and G acts continuously on K. We claim that there exists a Borel invariant probability measure \( \nu \) on X. Consider the continuous action of G on \( P(K) \) by \( g \mu(A) = \mu(g^{-1}A) \). By theorem 4.27 there exists an \( \alpha \)-invariant \( \mu \)-measurable map \( X \to P(K), x \mapsto \nu_x \). That is, \( \alpha(n, x) \nu_x = \nu_{nx} = \nu_{gx} \) with \( g = \alpha(n, x) \).

For \( A \subseteq X \) Borel, define

\[
\nu(A) = \int_X \nu_x(A) d\mu.
\]

Clearly \( \nu \) is a probability measure, so it is left to show it is invariant.
\[ v(gA) = \int_X v_x(gA) d\mu = \int_X g^{-1} v_x(A) d\mu = \int_X v_{g^{-1}x}(A) d\mu. \]

However the integral in the last line equals \( \int_X v_x(A) d\mu = v(A) \). So \( v \) is invariant as desired. \( \square \)

**Corollary 4.28.** The equivalence relation \( E(\mathbb{F}_2, 2) \) is not hyperfinite.

**Proof.** Let \( \mu \) be the usual product measure \( 2^{\mathbb{F}_2} \). Denote by \( (2)^{\mathbb{F}_2} \) the free part of the shift action \( \mathbb{F}_2 \curvearrowright 2^{\mathbb{F}_2} \). That is, \( (2)^{\mathbb{F}_2} = \{ x \in 2^{\mathbb{F}_2} : \forall g \neq 1 g \cdot x \neq x \} \). Clearly, \( \mathbb{F}_2 \) acts on \( (2)^{\mathbb{F}_2} \) freely and \( (2)^{\mathbb{F}_2} \) is a \( \mathbb{F}_2 \)-invariant Borel set. Let \( F(\mathbb{F}_2, 2) \) be the equivalence relation induced by the action \( \mathbb{F}_2 \curvearrowright (2)^{\mathbb{F}_2} \).

**Claim.** Let \( G \) be a countable group and \( \mu_G \) is the product measure on \( 2^G \), then \( \mu_G((2)^{\mathbb{F}_2}) = 1 \).

**Proof of the Claim.** Exercise.

Since \( \mu(\mathbb{F}_2) = 1 \), the restriction of \( \mu \) on \( (2)^{\mathbb{F}_2} \) is a probability measure. We know that \( \mathbb{F}_2 \) is not amenable. (See Proposition 4.21.) It follows by Theorem 4.25 that \( F(\mathbb{F}_2, 2) \) is not hyperfinite. Since hyperfinite equivalence relations are closed under restriction, \( E(\mathbb{F}_2, 2) \) is not hyperfinite either. \( \square \)

## 5 Treeability

Let \( E \) be a countable Borel equivalence relation on a standard Borel space \( X \). A **Borel graphing** of \( E \) is a Borel graph \( G = (X, R) \) whose connected components are exactly the \( E \)-classes. A **Borel treeing** of \( E \) is a graphing of \( E \) whose connected components are acyclic.

**Definition 5.1.** A Borel equivalence relation \( E \) is **treeable** if it admits a Borel treeing.

**Example 5.2.** Hyperfinite Borel equivalence relations are treeable.

**Proposition 5.3.** Let \( \mathbb{F}_n \) be the free group on \( n \) generators. Suppose that \( \mathbb{F}_n \) acts freely on a standard Borel space \( X \) in a Borel fashion. Then \( E_{\mathbb{F}_n}^X \) is treeable.
Proof. Let $\mathbb{F}_n$ be the free group with generators $(g_i)_{i < n}$ and let $E = E^X_{\mathbb{F}_n}$. For each $C \subseteq X / E$ and $x, y \in C$, define

$$(x, y) \in T \iff \exists i \ (g_i \cdot x = y \text{ or } x = g_i \cdot y).$$

A consequence of the previous proposition is that $F(\mathbb{F}_2, 2)$ is treeable. Hence the class of hyperfinite equivalence relations is strictly included in the treeable ones.

**Proposition 5.4.** Let $E$ be a countable Borel equivalence relation on a a standard Borel space $X$.

(i) If $E$ is treeable and $A \subseteq X$ is Borel, then $E \upharpoonright A$ is treeable.

(ii) If $A \subseteq X$ is a complete Borel section $E$ and $E \upharpoonright A$ is treeable, so is $E$.

(iii) If $E \leq_B F$ and $F$ is treeable, then so is $E$.

(iv) If $E \subseteq F$ and $F$ is treeable, so is $E$.

Proof. (i) Let $(E, T)$ be a treeing for $E$ with connected components $T_D$, for $D \in X / E$. Let $G = (g_i)_{i \in \mathbb{N}}$. For every $x \in X$, we consider the lexicographic order on the paths in $T_{[x]}$ starting at $x$ defined as follows:

$$(x, x_{1}, \ldots, x_{m}) <_{x} (x, x'_{1}, \ldots, x'_{n}) \iff m < n \text{ or }$$

$$\begin{cases} m = n \text{ and } \exists k \leq m \ ((x_{1}, \ldots, x_{k-1}) = (x'_{1}, \ldots, x'_{k-1}) \text{ and } \\
\exists i (g_{i}(x_{k-1}) = x_{k} \text{ and } \forall j < i (g_{j}(x'_{k-1}) \neq x'_{k})))\end{cases}$$

Now fix $C$ an equivalence class of $E \upharpoonright A$ and let $D = [C]_E$. For all $x \in D$ let $\rho(x)$ be the end of the $<_{x}$-least path from $x$ to (some point in) $C$. Now $\{\rho^{-1}(c)\}_{c \in C}$ is a partition of $D$. Note that if $\rho(x) = c$ and $x, x_1, \ldots, x_{m-1}, c$ is the $<_{x}$-least path from $x$ to $c$, then for each $k < m$, $x_k, \ldots, c$ is the $<_{x}$-least path from $x_k$ to $C$. So the restriction of $T_D \upharpoonright \rho^{-1}(c)$ is a tree for any $c \in C$. So $T_D$ induces a tree structure on the sets $(\rho^{-1}(c))_{c \in C}$. Last we define a tree structure $T_C$ on $C$ by setting:

$$(x, y) \in T_C \iff \exists x' \in \rho^{-1}(x), y' \in \rho^{-1}(y) \ ((x', y') \in T_D).$$

(ii) Let $f : X \to A$ be a Borel map with $f(x) = x$, and $f(x) = x'$ for $x \in A$. For each $C \in X / E$, let $T_{A \cap C}$ be the associated tree. We can define a tree $T_C$ from $T_{A \cap C}$ by adding an edge $(x, f(x))$ from each $x \in X \setminus A$.

The proof of (iii) follows by the previous items, while (iv) can be proved as (i).
Definition 5.5. Let \( X \) be a standard Borel space and \( E = E_G^X \) be a countable Borel equivalence relation. A Borel cocycle for \( E \) is a Borel map \( \rho: E \to G \) such that

\[
x \in E \, y \in E \, z \implies \rho(x, z) = \rho(y, z) \rho(x, y)
\]

Exercise 5.6. Suppose that \( \rho: E \to G \) be a Borel cocycle of \( E \). Then for all \( x, y \in X \),

(a) \( \rho(x, x) = 1 \);

(b) if \( x \in E \, y \) then \( \rho(x, y) = \rho(y, x)^{-1} \).

Definition 5.7 ([18]). An action \( a: G \acts X \) has the cocycle property if there is a Borel cocycle \( \rho: E \to G \) with \( \rho(x, y) \cdot x = y \).

If \( G \) acts on \( X \) freely then we can easily define Borel cocycle \( \rho: E_G \to G \). For each \( x \in E_G \, y \) we set \( \rho(x, y) \) equal to the unique \( g \in G \) such that \( g \cdot x = y \).

Theorem 5.8. For a countable Borel equivalence relation \( E \) the following are equivalent:

1. \( E \) is treeable;
2. For all countable \( G \) and for all \( G \)-action such that \( E = E_G^X \), this action has the cocycle property.
3. For all countable \( G \) such that \( E = E_G^X \) there is a free Borel action of \( G \) on some Borel Polish space \( Y \) such that \( E \cong_B E_G^Y \).

Proof. (3) \( \implies \) (1). Let \( E \) be a countable Borel equivalence on \( X \) satisfying (3). By Feldman-Moore theorem \( E = E_G^X \) for some countable \( G \) acting on \( X \). Since \( \mathbb{F}_\infty \) surjects onto any countable group, there is an action of \( \mathbb{F}_\infty \) on \( X \) such that \( E = E_{\mathbb{F}_\infty} \). Then, by the assumption, there exists some Polish space \( Y \) and some free Borel action \( \mathbb{F}_\infty \) so that \( E \cong_B Y_{\mathbb{F}_\infty} \). It follows that \( E \) is treeable.

(1) \( \implies \) (2). Assume that \( E \) induced by some Borel \( G \)-action and that \( E \) is treeable. Let \( (X, T) \) be a treeing for \( E \). Fix a Borel linear order \( \prec \) on \( X \), and put \( T^+ = T \cap \prec \). There exists a Borel function \( \rho^+: T^+ \to G \) such that for all \( x, y, \rho^+(x, y) \cdot x = y \). Next, define \( \rho: E \to G \) by setting:

(i) \( \rho(x, x) = 1 \);

(ii) If \( x \in E \, y \) and \( x = x_0 \cdot E x_n = y \) is the unique path from \( x \) to \( y \) in \( (X, T) \), put \( \rho(x, y) = \rho^+(x, x_n) \rho^+(x_{n-1}, x_n) \cdots \rho^+(x_0, x_1) \) where

\[
\rho^+(u, v) = \begin{cases} 
\rho^+(x, y) & \text{if } (x, y) \in T^+, \\
\rho^+(x, y)^{-1} & \text{if } (y, x) \in T^+.
\end{cases}
\]
(2) $\implies$ (3). Assume that $G$ is a countable group acting on $X$ in a Borel way so that $E = E_X^G$. Assuming (2) there exists a Borel cocycle $\rho: E \to G$ such that $\rho(x, y) \cdot x = y$ for all $x \in E \cdot y$. Now consider the action $G$ on itself $g \cdot h = gh$ by left multiplication. For $(x, g), (y, h) \in X \times G$ define

$$(x, g) \sim (y, h) \iff x \in y \text{ and } \rho(x, y)g = h.$$ 

Let $Y = (X \times G) / \sim$ with the quotient structure.

**Claim.** The relation $\sim$ is smooth, so $Y$ is standard. (Exercise) \hfill \square

Define an action $G \curvearrowright Y$ by $g \cdot [x, h] = [y_0, \rho(x, y_0)hg^{-1}]$ for any $y_0 \in [x]_E$. Notice that if $x \in y_0$, this is well-defined because $[y_0, \rho(x, y_0)hg^{-1}] = [x, hg^{-1}]$. Moreover, it is easily checked that the action is free.

\hfill \square

**Theorem 5.9** ([20]). If $E$ is treeable, then $E \leq_B F(\mathbb{F}_2, 2)$. Hence $F(\mathbb{F}_2, 2)$ is the universal treeable equivalence relation.

**Question 5.10** (The finite index problem). Let $E \subseteq F$ be countable Borel equivalence relations such that $E$ is treeable and every $F$-class contains only finitely many $E$-class. Is $F$ treeable?

For some groups there are intrinsic restriction to induce a treeable equivalence relation. Those groups are called antitreeable.

**Theorem 5.11** (Adams-Spatzier [2]). If $G$ is an infinite Kazhdan group acting on $X$ so that $E = E_X^G$ and there is an $E$ invariant probability measure on $X$, then $E$ is not treeable.

Examples of Kazhdan groups include the special linear groups $\text{SL}_n(\mathbb{Z})$ and the projective special linear group $\text{PSL}_n(\mathbb{Z})$, which is the quotient of $\text{SL}_n(\mathbb{Z})$ modulo its center, for $n \geq 3$. (E.g., see Lubotzky [28, Chapter 3].)

Let $R_n$ be the equivalence relation induced from the action of $\text{GL}_n(\mathbb{Z})$ on $\mathbb{T}^n$ by $$(a_{ij}) \cdot (x_1, \ldots, x_n) = (\sum_j a_{1j}x_j, \ldots, \sum_j a_{nj}x_j).$$

**Corollary 5.12.** The equivalence relation $R_n$ is not treeable for $n \geq 3$. 

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6 Torsion-free abelian group of finite rank

Recall that an abelian group $A$ is torsion-free if every nonzero element has infinite order, i.e., $na \neq 0$ for all $a \in A \setminus \{0\}$ and $n \in \mathbb{N}$.

Let $A$ be a countable abelian group, and $\mathbb{P}$ be the set of prime numbers.

**Definition 6.1.** The $p$-height of $a$ is defined as

$$h_a(p) = \sup\{k \in \mathbb{N} : \text{there is } x \in A \text{ such that } p^k x = a \in \mathbb{N} \cup \{\infty\}\}.$$

Then the characteristic (or height sequence) $\chi(a)$ of $a$ is defined to be

$$\chi(a) = (h_a(p) \mid p \in \mathbb{P}).$$

**Fact 6.2.**

(i) $\chi(a) = \chi(-a)$;

(ii) If $\chi(a) = (k_{p_1}, \ldots, k_{p_n}, \ldots)$ then $a$ is divisible $m = \prod_{i \in I} p_i^{\ell_i}$ if and only if $\ell_i \leq k_{p_i}$ for $i \in I$.

(iii) $\chi(p^na) = (h_a(p_1), \ldots, h_a(p_n) + 1, \ldots)$. For this to make sense set $\infty + 1 = \infty$.

(iv) Any sequence $\chi \in (\mathbb{N} \cup \{\infty\})^\mathbb{P}$ can be realized as the height sequence of some elements in some torsion-free abelian group. In fact, if $A = \langle \frac{1}{p_n^\ell_n} \mid \ell_n \in \mathbb{N}, \ell_n \leq \chi(p_n) \rangle$ is a subgroup of $\mathbb{Q}$ such that $\chi_A(1) = (\chi(p_1), \chi(p_1), \ldots, \chi(p_n), \ldots)$.

**Definition 6.3.** Two $h$-sequences $\chi_1, \chi_2 \in (\mathbb{N} \cup \{\infty\})^\mathbb{P}$ belong to the if and only if

(a) $\chi_1(p) = \chi_2(p)$ for almost all primes.

(b) $\chi_1(p) \neq \chi_2(p)$, then both $\chi_1(p)$ and $\chi_2(p)$ are finite.

We define the type $t(a)$ of $a$ as the class $[\chi(a)]_\equiv$.

**Lemma 6.4.**

(a) If $a$ and $b$ are linearly dependent, then $t(a) = t(b)$.

(b) If $h : A \to B$ is a group homomorphism, then $t(a) \leq t(h(a))$.

**Proof.** To see (a) notice that $t(a) = t(ma)$ for all $m \in \mathbb{N}$. Thus if $ma = nb$ for some $m, n \in \mathbb{Z}$, then

$$t(a) = t(ma) = t(na) = t(b).$$

\[\square\]
Recall that an abelian group has rank 1 if and only if every two nonzero element are linearly dependent. Hence we can define the type $t(A)$ of $A$ to be the $\equiv$-equivalence class containing $\chi(a)$, where $a$ is any non-zero element of $A$.

**Theorem 6.5** (Baer [3]). Two torsion-free abelian groups of rank 1 are isomorphic if and only if they have the same type.

**Proof.** If $a \in A$ and $b \in B$ and $t(a) = t(b)$ then there exist integers $m, n$ such that $\chi(ma) = \chi(nb)$. Then we can extend the partial function $ma \mapsto nb$ to an isomorphism $A \to B$. \qed

**Definition 6.6.** Let $S(Q) \subseteq 2^Q$ be the Polish space of torsion-free abelian group of rank 1. ($S(Q) = \{X \subseteq Q : X$ is a subgroup of $Q\}$ is closed in $Q$ so $S(Q)$ is a (compact) Polish space.)

The following is essentially a reformulation of Theorem 6.5.

**Proposition 6.7.** $\cong_1$ is Borel equivalent to $E_0$.

Fuchs [12, Question 66] asked for a characterization of the invariant for the rank 2 torsion-free abelian groups. In order to address the question of Fuchs using techniques from the theory of countable Borel equivalence relation we introduce the relevant Borel spaces of rank 2 torsion-free abelian groups.

Let $S(Q^n) = \{A \subseteq Q^n : A \leq Q^n\}$ and $\cong_n$ be the isomorphism relation on $S(Q^n)$. As usual, denote by $GL_n(Q)$ the countable group of all invertible matrices $n \times n$ with entries in $Q$. It is clear that $GL_n(Q)$ acts on $S(Q^2)$ by linear transformation and $\cong_n$ coincide with the orbit equivalence relation arising from this action, i.e., for $A, B \in S(Q^n)$

$$A \cong_n B \iff \exists M \in GL_n(Q) \ (M \cdot A = B).$$

Next theorem shows that such invariant are genuinely more complicated than the ones introduced by Baer in the case of torsion-free abelian groups of rank 1.

**Theorem 6.8** (Hjorth [16]). $\cong_2$ is not hyperfinite. Hence, $\cong_1 <_B \cong_2$.

We consider a subrelation of $\cong_2$ restricted to a subspace of $S(Q^2)$

Since $SL_2(Z^2) \subseteq GL_2(Q^2)$ preserves $Z^2$, it acts on $S(Q^2, Z^2) = \{A \in S(Q^2) : Z^2 \subseteq A\}$.

Let $S(Q^2/Z^2) = \{A \subseteq Q^2/Z^2 : A \leq Q^2/Z^2\}$ be the standard Borel space of subgroups of $Q^2/Z^2$. We consider the projective special linear group $PSL_2(Z) = SL_2(Z)/\{1, -1\}$. Since $SL_2(Z)$ is not amenable and $\{1, -1\}$ is obviously amenable, $PSL_2(Z)$ is not amenable either. Instead of
considering the action of $\text{GL}_2(\mathbb{Z})$ we consider the equivalence subrelation induced by the action of 
$\text{PSL}_2(\mathbb{Z})$. (Notice that the $\text{SL}_2(\mathbb{Z})$ pushes down to a $\text{PSL}_2(\mathbb{Z})$-action because $1 \cdot A = -1 \cdot A = A$.)

Let $\pi_2: \mathbb{Q}^2 \to \mathbb{Q}^2/\mathbb{Z}^2$ be the canonical projection of $\mathbb{Q}^2$ onto $\mathbb{Q}^2/\mathbb{Z}^2$. The $\text{PSL}_2(\mathbb{Z})$-action on $S(\mathbb{Q}^2, \mathbb{Z}^2)$ projects down to a $\text{PSL}_2(\mathbb{Z})$-action on $S(\mathbb{Q}^2/\mathbb{Z}^2)$.

For ease of exposition let $\text{PSL} = \text{PSL}_2(\mathbb{Z})$ and let $S_2$ be $S(\mathbb{Q}^2/\mathbb{Z}^2)$. We are going to show that $E_{\text{PSL}}$ is not hyperfinite. Note that this will imply that $E_{\text{PSL}}$ is not hyperfinite as the point-wise image map $\pi_2[-]: S(\mathbb{Q}^2) \to S(\mathbb{Q}^2/\mathbb{Z}^2)$ is a bijection. It will follow that $\sim_2$ is not hyperfinite.

Let $\Gamma = \widehat{\mathbb{Q}^2/\mathbb{Z}^2} = \text{Hom}(\mathbb{Q}^2/\mathbb{Z}^2, \mathbb{T})$ be the dual of $\mathbb{Q}^2/\mathbb{Z}^2$. We regard $\Gamma$ with the point-wise addition and the induced topology from $\mathbb{T}^{\mathbb{Q}^2/\mathbb{Z}^2}$. $\Gamma$ is a compact Polish group. So we apply a classical theorem of Haar.

**Theorem 6.9 (Haar).** Every locally compact Hausdorff topological group admits a unique, up to a constant multiple, nontrivial left-invariant measure that is finite on compact sets.

Let $\mu$ be the Haar probability measure on $\Gamma$. Then let $\nu$ be the pushforward measure along the map $\ker: \psi \mapsto \ker(\psi)$. That is $\nu(B) = \mu(\ker^{-1}[B])$.

**Lemma 6.10.** $\nu$ is $\text{SL}_2(\mathbb{Z})$-invariant.

**Proof.** Fix $M \in \text{SL}_2(\mathbb{Z})$.

\[
\nu(M \cdot A) = \mu(\ker^{-1}[M \cdot A]) \\
= \mu(\{ \gamma \in \Gamma : \ker(\gamma) \in M \cdot A \}) \\
= \mu(\{ \gamma \in \Gamma : \{ g : \gamma(g) = 0 \} \in M \cdot A \}) \\
= \mu(\{ \gamma \in \Gamma : \{ M^{-1}g : \gamma(g) = 0 \} \in A \}) \\
= \mu(\{ \gamma \in \Gamma : \{ g : \gamma(Mg) = 0 \} \in A \}).
\]

Notice that the map $\rho_M: \Gamma_2 \to \Gamma_2$, by $\rho_M(\gamma)(x) = \gamma(M^{-1} \cdot x)$ is a homomorphic homeomorphism. So the last line of previous chain of equality is the same as

\[
\mu(\{ \gamma \in \Gamma : \ker(\rho_M^{-1}(\gamma)) \in A \}) \\
= \mu(\rho_M[\ker^{-1}[B]]) = \mu(\ker^{-1}[B]) = \nu(B).
\]

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Lemma 6.11. There is an invariant measure 1 on which $\text{PSL}_2(\mathbb{Z})$ acts freely.

Proof. See [16, Lemma 5.3]

Proof of Theorem 6.12. Since $\text{PSL}_2(\mathbb{Z})$ is not amenable, $\nu$ is a $\text{PSL}_2(\mathbb{Z})$-invariant probability measure, and $\text{PSL}_2(\mathbb{Z})$ acts freely on a measure 1 invariant subset of $S_2$, it follows by Theorem 4.25 that $E_p^{S_2} SL$ is not hyperfinite. Therefore, $\cong_2$ contains a non-hyperfinite equivalence relation and is not hyperfinite either.

Notice that every free action of $\text{PSL}_2(\mathbb{Z})$ induces a treeable equivalence relation. Hence our method does not say anything about the treeability of $\cong_2$. In a similar way the following is a consequence of anti-treeability (see Theorem 5.11) and the fact that $\text{SL}_n(\mathbb{Z})$ is Kazhdan for all $n \geq 3$.

Theorem 6.12 (Hjorth [16]). $\cong_n$ is not treeable.

Definition 6.13. Let $G$ be a locally compact second countable group. A subgroup $H \subseteq G$ is said to be a lattice in $G$ if it is discrete and there is an invariant Borel probability measure for the action of $G$ on the cosets $G/H$ by $g \cdot hH = ghH$.

Kechris sharpened Hjorth’s results by showing that $\cong_2$ is not treeable. In his proof he presented $\text{PSL}_2(\mathbb{Z}[1/2])$ as a lattice in a product group $\text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{Q}_2)$ and lifted a $\text{PSL}_2(\mathbb{Z}[1/2])$-invariant probability measure to a measure invariant under the action of the product group. Note that both $\text{PSL}_2(\mathbb{R})$ and $\text{PSL}_2(\mathbb{Q}_2)$ contain an infinite amenable discrete group. E.g., consider $
\begin{bmatrix} 2^k & 0 \\ 0 & 2^{-k} \end{bmatrix} \mid k \in \mathbb{Z}\n$

Theorem 6.14 (Kechris [23]). Suppose that $H$ is a countable group which is a lattice in a product $G_1 \times G_2$ of two locally compact second countable groups, each of which contains an infinite amenable discrete subgroup. If $H$ is not amenable and acts freely on a standard Borel space $X$ with an invariant probability measure, then $E_H^X$ is not treeable.

7 Popa’s cocycle superrigidity

The cocycle superrigidity theorem of Popa is an instance of cocycle reduction result. We briefly discuss the idea behind it.
First recall the notions of cocycle for Borel equivalence relations and group actions. The main difference from previous chapters is that we will be working in the measure theoretical setting. Until further notice let $\mu$ be a $G$-invariant probability measure on $X$.

**Definition 7.1.** Let $E$ be a countable Borel equivalence relation on $X$. A map $\rho: E \to G$ is a cotcycle if it satisfies the cocycle identity

$$\rho(x, z) = \rho(y, z)\rho(x, y)$$

for $\mu$-a.e. $x, y \in X$.

**Definition 7.2.** Two Borel cocycles $\rho, \sigma: E \to G$ are equivalent (in symbols $\rho \sim \sigma$) if there is a Borel map $f: X \to G$ such that

$$x E y \implies \sigma(x, y) = f(y)\rho(x, y)f(x)^{-1}$$

for $\mu$-a.e. $x, y \in X$.

**Definition 7.3.** Let $G$ be a countable group and $a: G \actson X, a(g, x) = g \cdot x$ be a Borel action. A Borel cocycle for $a$ into $H$ is a map $\alpha: G \times X \to H$ such that for all $g_0, g_1 \in G$,

$$\alpha(g_1g_0, x) = \alpha(g_1, g_0 \cdot x)\alpha(g_0, x)$$

for $\mu$-a.e. $x, y \in X$.

**Example 7.4.** Let $a: G \actson X$ be a Borel action and $h: G \to H$ be a group homomorphism. Then we can simply define a cocycle by $\alpha(g, x) = h(g)$ for all $x \in X$.

**Remark 7.5.** If $a: G \actson X$ is a Borel action. Any cocycle $\rho: E_G \to H$ induces a cocycle $\alpha: G \times X \to H$ for $a$ by setting

$$\alpha(g, x) = \rho(x, g \cdot x).$$

If $a: G \to X$ is a free action, then any cocycle $\alpha: G \times X \to H$ gives rise to a cocycle $\rho: E_G \to H$ by setting $\rho(x, g \cdot x) = \alpha(g, x)$.

**Definition 7.6.** Two Borel cocycles $\alpha, \beta: G \times X \to H$ for $a$ are equivalent (in symbols $\alpha \sim \beta$), if there is a Borel map $f: X \to H$ such that

$$\beta(g, x) = f(g \cdot x)\alpha(x, y)f(x)^{-1}$$

for $\mu$-a.e. $x, y \in X$.

We are now ready to state a special case of Popa’s superrigidity cocycle stating that under certain hypotheses on a group $G$ acting on the space $(X, \mu)$, every Borel cocycle from $G \times X$ into an arbitrary countable group is equivalent to a group homomorphism. (A self-contained purely ergodic-theoretic presentation of Popa’s proof was given by Furman [13].)
Theorem 7.7 (Popa [37]). Let $G \leq G$ be countable groups such that $G$ contains a Kazhdan infinite normal subgroup. If $H$ is any countable group, then every Borel cocycle

$$\alpha: G \times (2)^G \to H$$

for the shift action of $G$ on $(2)^G$ is equivalent to a group homomorphism $G \to H$.

In most applications we can let $G = G$. This will not be the case in the proof of Theorem 9.10.

In order to explain how 7.7 can be used, consider the following situation. Let $a: G \curvearrowright X$ and $b: H \curvearrowright Y$ be Borel action on the standard Borel spaces $X$ and $Y$ respectively, and suppose that $b$ is free. Put $E = E_a$ and $F = E_b$. Moreover, assume that $\varphi: X \to Y$ is a Borel homomorphism from $E$ to $F$ and $a: G \times X \to H$ is the Borel cocycle of $a$ associated to $\varphi$. That is, for every $(g, x) \in G \times X$ there is a unique $a(g, x) \in H$ such that $\varphi(g \cdot x) = a(g, x) \cdot \varphi(x)$. If $a \sim \beta$ via $f: X \to H$, the perturbation $\varphi'(x) = f(x)\varphi(x)$ of $\varphi$ is also a homomorphism from $E$ to $F$. Moreover, $\beta$ is the cocycle associated to $\varphi'$. That is, $\varphi'(g \cdot x) = \beta(g, x) \cdot \varphi'(x)$.

Now if any cocycle $a: G \times X \to Y$ for $a$ is equivalent to a group homomorphism $h: G \to H$ and $\varphi': X \to Y$ is the homomorphism from $E$ to $F$ associated to $h$, then we have

$$\varphi'(g \cdot x) = h(g) \cdot \varphi'(x)$$

(†)

If in a certain situation we can exclude that such $\varphi'$ as in (†) exists, then we can exclude the existence of a Borel homomorphism from $E$ to $F$ at all. This will show that $E$ is not Borel reducible to $F$ in a very strong sense.

Applications of superrigidity

Let $G$ be a countable acting on a standard Borel space $X$ in a Borel fashion. Suppose that $\mu$ is a $G$-invariant probability measure on $X$.

Definition 7.8. We say that the action of $G$ on $(X, \mu)$ is ergodic (or $G$ acts ergodically) if for every $G$-invariant Borel $A \subseteq X$ either $\mu(A) = 1$ or $\mu(A) = 0$.

The following is a characterization of ergodic actions. For the proof and a more general statement see [4, Theorem 1.3].

Proposition 7.9. For a locally compact second countable topological group $G$ the following are equivalent:

(i) $G$ acts on $(X, \mu)$ ergodically;
(ii) If $Y$ is a standard Borel space and $f : X \to Y$ is a $G$-invariant map (i.e., $f(g \cdot x) = f(x)$), then there is $X_0 \subseteq X$ with $\mu(X_0) = 1$ such that $f \upharpoonright X_0$ is constant.

We introduce a stronger notion of ergodicity.

**Definition 7.10.** We say that the action of $G$ on $(X, \mu)$ is **strongly mixing** if and only if for any two Borel subsets $A, B \subseteq X$, if $(g_n)_{n \in \mathbb{N}}$ is a sequence of distinct elements of $G$, then

$$\lim_{n \to \infty} \mu(g_n(A) \cap B) = \mu(A)\mu(B).$$

Notice that any strongly mixing action is necessarily ergodic. To see this, let $A$ be a $G$-invariant Borel subset of $X$. Then $g(A) \cap A = A$ for all $g \in G$. Hence for any sequence $(g_n)_{n \in \mathbb{N}}$ of distinct elements of $G$,

$$\mu(A) = \lim_{n \to \infty} \mu(A) = \lim_{n \to \infty} \mu(g_n(A) \cap A) = \mu(A)\mu(A),$$

therefore $\mu(A)$ is either 0 or 1.

**Lemma 7.11.** If $G$ is a countable infinite group and $\mu$ is the product measure on $(2)^G$, then the action of $G$ on $((2)^G, \mu)$ is strongly mixing.

**Proof.** See [19, Proposition A6.1].

Note that if the action of $G$ on $(X, \mu)$ is strongly mixing and $H$ is an infinite subgroup of $G$, then the action of $H$ is also strongly mixing, in particular $H$ acts ergodically on $(X, \mu)$.

**Definition 7.12.** A Borel homomorphism from $E$ to $F$ is $\mu$-trivial if there is a set $X_0 \subseteq X$ with $\mu(X_0) = 1$ such that $f$ maps $X_0$ to a single $F$-class.

**Definition 7.13.** A **virtual embedding** between groups $h : G \to H$ is a homomorphism $h : G \to H$ with $|\ker h| < \infty$.

**Theorem 7.14.** Let $G = \text{SL}_3(\mathbb{Z}) \times S$, where $S$ is any countable group. Suppose that $H$ is any countable group and that $Y$ is a free standard Borel $H$-space. If there exists a $\mu$-nontrivial Borel homomorphism from $F(G, 2)$ to $E_H^1$, then there exists a virtual embedding $h : G \to H$.

**Proof.** Let $G = \text{SL}_3(\mathbb{Z}) \times S$ and let $H$ act freely on $Y$. Let $E = F(2, G)$ and $F = E_H$. Suppose that $\varphi : (2)^G \to Y$ is a $\mu$-nontrivial Borel homomorphism from $E$ to $F$. Then we can define a Borel cocycle $\alpha : G \times X \to H$ by letting $\alpha(g, x)$ be the unique element of $H$ such that

$$\varphi(g \cdot x) = \alpha(g, x) \cdot \varphi(x).$$
By Theorem 7.7 we can suppose that $\alpha : G \to H$ is a homomorphism, after adjusting $\phi$ if necessary. Now suppose for contradiction that $K = \ker \alpha$ is infinite. Since the action of $G$ on $(2)^G$ is strongly mixing, it is clear that $K$ acts on $(2)^G$ ergodically. Moreover notice that the function $\phi : (2)^G \to Y$ is a $K$-invariant function. Thus, it follows that $\phi$ is constant $\mu$-almost everywhere by Proposition 7.9. And this contradicts the assumption on $\phi$. 

**Definition 7.15.** A countable Borel equivalence relation $E$ is free if there exists a countable group $G$ acting on $X$ freely such that $E = E_G^X$.

**Theorem 7.16** (Thomas [42]). There are continuum many pair-wise incomparable free countable Borel equivalence relations

**Proof.** Let $\mathbb{P}$ be the set of primes and for each prime $p \in \mathbb{P}$, let $A_p = \bigoplus_{i=0}^{\infty} C_p$ be the direct sum of countably many copies of the cyclic group $C_p$ of order $p$. For each subset $C \subseteq \mathbb{P}$, let $G_C = SL_3(\mathbb{Z}) \times \bigoplus_{p \in C} A_p$.

**Claim 7.16.1.** $C \subseteq D$ if and only if $F(G_C, 2) \leq_B F(G_D, 2)$.

If $C \subseteq D$, then $G_C \leq G_D$ and hence $F(G_C, 2) \leq_B F(G_D, 2)$. Conversely, if $F(G_C, 2) \leq_B F(G_D, 2)$, then there exists a virtual embedding $h : G_C \to G_D$. It is well-known that $SL_3(\mathbb{Z})$ contains a torsion-free subgroup of finite index. (E.g., see Wehrfritz [49, Corollary 4.8].) It follows that for each $p \in C$, the cyclic group $C_p$ embeds into $\bigoplus_{q \in D} A_q$ and this can be possible only if $p \in D$. 

**Definition 7.17.** A countable Borel equivalence relation $E$ is essentially free if there exists a free countable Borel equivalence relation such that $E \sim_B E_G^X$.

**Theorem 7.18** (Thomas [42]). If $E$ is essentially free countable Borel equivalence relation, then there exists a countable group $G$ such that $F(G, 2) \not\leq_B E$.

**Proof.** Let $H$ be a countable group. Suppose that $E \sim_B E_H^Y$ for some free action of $H$ on a standard Borel space $Y$. Let $L$ be any finitely generated group that does not embed into $H$. (This existence of such group is guaranteed by a classical result of Neumann [34, Theorem 14] stating that there are uncountably many finitely generated group.) Let $S = L \ast \mathbb{Z}$ and $G = SL_3(\mathbb{Z}) \times S$. If $F(G, 2) \leq E$ then there is a virtual embedding $\pi : G \to H$. Then $\pi \upharpoonright S$ is a virtual embedding too. However $L \ast \mathbb{Z}$ contains no normal nontrivial subgroup, therefore $|\ker \pi \upharpoonright S| = 1$. This implies that $L$ embeds into $H$ which is impossible. 

$\square$
Corollary 7.19. The class of essentially free countable Borel equivalence relations does not admit a universal element. In particular, \( E(\mathbb{F}_2, 2) \) is not essentially free.

We end this section stating other applications of Popa superrigidity theorem for countable Borel equivalence relations.

Theorem 7.20 (Thomas [42]). There exist uncountably many non-essentially free countable Borel equivalence relations that are incomparable up to Borel reducibility.

Theorem 7.21 (Thomas [42]). Suppose that \( G \) is a quasi-finite group. Then \( E_G^X \) is not universal.

Definition 7.22. Suppose that \( E \) and \( F \) are countable Borel equivalence relations on the standard Borel spaces \( X \) and \( Y \), respectively. We say that \( E \) is weakly Borel reducible to \( F \) if there is a countable-to-one Borel homomorphism \( f : X \to Y \) from \( E \) to \( F \).

Theorem 7.23 (Thomas [42], after Adams [1] and Horth-Kechris [19]). There is a pair of countable Borel equivalence relation such that \( E \subseteq F \) and \( E \) and \( F \) are incomparable with respect to Borel reducibility.

Theorem 7.24 (Thomas [42]). If \( E \) is a weakly universal countable Borel equivalence relation, then \( E \) is not essentially free.

Proof. Otherwise, we can suppose that \( E = E_H^Y \) for some free Borel action \( H \acts Y \). But then there exists a countable \( G \) such that every homomorphism from \( F(G, 2) \) to \( E_H^X \) maps a measure one set into a single \( E_H^Y \)-class. Therefore, \( E_G \) is not weakly Borel reducible to \( E_H^Y \).

\( \square \)

8 (Anti)classification of torsion-free abelian groups of finite rank

In this section we discuss the proof of the following result:

Theorem 8.1 (Thomas [41]). For all \( n \geq 3 \),

\[
\cong_{n-1} <_B \cong_n.
\]

Theorem 8.1 is a consequence of the following result.

Theorem 8.2 (Thomas [41]). Let \( n \geq 3 \). Suppose that \( \text{SL}_n(\mathbb{Z}) \) acts on a standard Borel space \( X \) and \( \mu \) is an ergodic \( \text{SL}_n(\mathbb{Z}) \)-invariant probability measure. If \( 1 \leq m < n \) and \( f : X \to S(\mathbb{Q}^m), x \mapsto A_x \) is a Borel homomorphism from \( E_{\text{SL}_n(\mathbb{Z})}^X \) to \( \cong_m \), then there exists a \( \text{SL}_n(\mathbb{Z}) \)-invariant Borel \( M \subseteq X \) such that \( f \) maps \( M \) into a single \( \cong_m \)-class.
Exercise 8.3. Suppose that a countable group $G$ is acting on a standard Borel space $X$ in a Borel fashion and $\mu$ is an ergodic $G$-invariant probability measure on $X$. If $Y$ is a a standard Borel space, $H$ is a countable group acting on $Y$ in a Borel fashion, $f : X \to Y$ is a Borel homomorphism from $E^X_G$ to $E^Y_H$, and $\{\text{Stab}_H(f(x)) : x \in X\}$ is countable. Then there exists a fixed subgroup $L \leq G$, a $G$-invariant $X_1 \subseteq X$ with $\mu(X_1) = 1$, and a Borel homomorphism $f' : X_1 \to Y$ from $E^{X_1}_G$ to $E_H$ such that $\text{Stab}_H(f'(x)) = L$, for all $x \in X_1$.

In the scenario described by Exercise 8.3 the equivalence relation $E_H \upharpoonright f(X_1)$ will be induced by a free action of the quotient group $H = N_H(L)/L$ on the Borel set $Y = \{y \in Y : \text{Stab}(y) = L\}$. This will allow us to define a cocycle $\alpha : G \times X_1 \to N_H(L)/L$ where $N_H(L) = \{h \in H : hL = Lh\}$ is the normalizer of $L$ in $H$. So we might be able to apply a suitable cocycle reduction theorem. Unfortunately, the situation of Theorem 8.2 is way different than the idyllic landscape described above. The isomorphism relation $\cong_m$ is induced by the action of $\text{GL}_m(\mathbb{Q})$ and unfortunately there are uncountably possibilities for $\text{Stab}(A_x) = \text{Aut}(A_x)$. This explains why the proof of Theorem 8.2 is quite involved. To come across this difficulty we shall consider a coarser equivalence relation on $S(\mathbb{Q}^n)$.

Quasi-equality and quasi-isomorphism

Definition 8.4. Suppose that $A, B \in S(\mathbb{Q}^n)$. We say that $A$ and $B$ are quasi-equal (in symbols $A \approx_m B$) if and only if $A \cap B$ has finite index in both $A$ and $B$.

Definition 8.5. Suppose that $A, B \in S(\mathbb{Q}^n)$. We say that $A$ and $B$ are quasi-isomorphic (in symbols $A \sim_m B$) if and only if there is $\varphi \in \text{GL}_n(\mathbb{Q})$ such that $\varphi \cdot A \approx_m B$.

Proposition 8.6. For all $n > 0$, $\approx_n$ is a countable Borel equivalence relation. Then, so is $\sim_n$.

Proof. Given $A \in S(\mathbb{Q}^n)$ there are only countably many $B \in S(\mathbb{Q}^n)$ such that $A \approx_n B$. To see this, notice that if $A \approx_n B$ then there are finite integers $r, s > 0$ such that $rA \leq B$ and $sB \leq A$. Then $B$ satisfies necessarily

$$rA \leq B \leq \frac{1}{s}A$$

and $[\frac{1}{s}A : rA] = [A : rsA] < \infty$. There are only countably many possibilities for $B$. \qed

In the proof of Theorem 8.2 we shall consider the action of $\text{GL}_m(\mathbb{Q})$ on the $\approx_m$-classes. First, we need to give a characterization of the stabilizer for this action.
Definition 8.7. A linear transformation $\varphi \in \text{Mat}_m(\mathbb{Q})$ is a quasi-endomorphism of $A$ if and only if $\varphi(A) \prec_m A$. Equivalently, if and only if there exists $k > 0$ such that $k\varphi \in \text{End}(A)$.

Fact 8.8. Let $A, B \in S(Q^m)$.

- $\text{End}(A)$ is a $\mathbb{Q}$-subalgebra of $\text{Mat}_m(\mathbb{Q})$.
- If $A \approx_m B$, then $\text{QE}(A) = \text{QE}(B)$.

Definition 8.9. A linear transformation $\varphi \in \text{Mat}_m(\mathbb{Q})$ is a quasi-automorphism of $A$ if and only if $\varphi$ is a unit of the $\mathbb{Q}$-algebra $\text{QE}(A)$.

Fact 8.10. Let $A \in S(Q^m)$ and $\varphi \in \text{End}(A)$

- $\text{QAut}(A)$ is a group,
- $\varphi \in \text{QAut}(A)$ if and only if $\varphi$ is a monomorphism.

Lemma 8.11 (Thomas [41, Lemma 3.6]). If $A \in S(Q^m)$, then $\text{QAut}(A)$ is the (set-wise) stabilizer of $[A]$ in $\text{GL}_n(\mathbb{Q})$.

Proof of Theorem 8.2

Clearly $x E y$ implies $A_x \sim_m A_y$, hence $f$ is a Borel homomorphism from $E$ to $\sim_m$. By Lemma 8.11 it follows that $|\{\text{QE}(A_x) : x \in X\}| \leq \omega$ because there are only countably many possibilities for the $\mathbb{Q}$-subalgebras of $\text{Mat}_m(\mathbb{Q})$.

Therefore there is a $\text{SL}_n(\mathbb{Z})$-invariant Borel subset $X_1 \subseteq X$ with $\mu(X_1) = 1$ and after possibly adjusting $f$ there is a fixed $\mathbb{Q}$-subalgebra $S$ of $\text{Mat}_m(\mathbb{Q})$ such that $\text{QE}(A_x) = S$ for all $x \in X_1$. For the sake of readability we put $X = X_1$. In particular, for all $X = X_1$, $\text{QAut}(A_x) = S^*$ the group of units of $S$.

Now, for all $x, y \in X$ such that $x E y$, $A_x \approx A_y$. So there is $\varphi \in \text{GL}_n(\mathbb{Q})$ such that $\varphi(A_x) \approx_m A_y$.

Note that $\varphi \in N_{\text{GL}_n(\mathbb{Q})}(S)$ as

$$\varphi S \varphi^{-1} = \varphi \text{QE}(A_x) \varphi^{-1} = \text{QE}(\varphi(A_x)) = \text{QE}(\varphi(A_y)) = S.$$ 

Let $H = N_{\text{GL}_n(\mathbb{Q})}(S)/S^* = \{\varphi S^* : \varphi \in N_{\text{GL}_n(\mathbb{Q})}\}$. For readability let $\bar{\varphi} = \varphi S^*$. Notice that $H$ acts on $f(X_1)/\sim_m$ freely by $\bar{\varphi} \cdot [A] = [\varphi(A)]$. (To see this, observe that $\varphi([A]) = \bar{\varphi} \cdot [A] = [A]$ implies that $\varphi \in \text{QAut}(A)$ by Lemma 8.11, thus $\varphi \in S^*$.)

Then we can define a Borel cocycle $\alpha : \text{SL}_n(\mathbb{Z}) \times X \to H$ by letting $\alpha(g, x)$ be the unique $\bar{\varphi} \in H$ such that $\bar{\varphi} \cdot [A_x] = [A_{g,x}]$. Now we are ready to apply the following reduction theorem:
Theorem 8.12 (Thomas [41]). Let \( n \geq 3 \). Suppose that \( \text{SL}_n(\mathbb{Z}) \) acts on a standard Borel space \( X \) and \( \mu \) is an ergodic \( \text{SL}_n(\mathbb{Z}) \)-invariant probability measure. If \( G \) is an algebraic \( \mathbb{Q} \)-group with \( \dim G < n^2 - 1 \) and \( H \leq G(\mathbb{Q}) \). Then every Borel cocycle \( \alpha : \text{SL}_n(\mathbb{Z}) \times X \to H \) is equivalent to a Borel cocycle \( \beta \) such that \( \beta(\text{SL}_n(\mathbb{Q}) \times X) \) is contained in a finite subgroup of \( H \).

The fact that such algebraic \( \mathbb{Q} \)-group exists is nontrivial (see [41, Lemma 3.7] for details).

So let \( \beta : \text{SL}_n(\mathbb{Z}) \times X \to H \) a cocycle such that \( \beta(\text{SL}_n(\mathbb{Z}) \times X) \) is contained in a finite subgroup \( K < H \). Also let \( b : X \to H \) be a Borel map such that for all \( g \in \text{SL}_n(\mathbb{Z}) \)

\[
\alpha(g,x) = b(g \cdot x)\beta(g,x)b(x)^{-1} \quad \text{for } \mu\text{-a.e. } x \in X.
\]

Theorem 8.13 (Hjorth-Kecris [18, Theorem 10.5]). Suppose that a countable infinite Kazhdan group \( G \) acts on a standard Borel space \( X \) and \( \mu \) is an ergodic \( G \)-invariant probability measure. If \( F \) is any treeable equivalence relation on \( Y \) and \( f \) is a Borel homomorphism from \( E^X_G \) to \( F \) then there exists a Borel \( X_0 \subseteq X \) such that \( \mu(X_0) = 1 \) and \( f \) maps \( X_0 \) into a unique \( F \)-class.

For all \( k > 2 \), and \((A_1, \ldots, A_k), (B_1, \ldots, B_k) \in (S(\mathbb{Q}))^k\) define

\[
(A_1, \ldots, A_k) F (B_1, \ldots, B_k) \iff \{[A_1], \ldots, [A_k]\} = \{[B_1], \ldots, [B_k]\}.
\]

Theorem 8.14 (Thomas [41, Theorem 3.8]). For all \( n > 0 \), the countable Borel equivalence relation \( \approx_n \) is hyperfinite.

Corollary 8.15. For all \( n > 0 \), \( F_n \) is hyperfinite.

Lemma 8.16. There exist a \( \text{SL}_n(\mathbb{Z}) \)-invariant \( X_1 \subseteq X \) with \( \mu(X_1) = 1 \), a fixed \( k > 0 \), and a Borel homomorphism \( \bar{f} : X_1 \to (S(\mathbb{Q}^m))^k \), from \( E^Z_{\text{SL}_n(\mathbb{Z})} \) to \( F_k \) such that for every \( x \), the groups in \( \bar{f}(x) \) are all quasi-isomorphic to \( A_x \).

Lemma 8.16 together with theorem 8.13 gives that \( \bar{f} \) maps \( X_1 \) into a unique \( F_k \)-class. Therefore the is a \( \approx_m \)-class \( C \subseteq S(\mathbb{Q}^m) \) such that \( f(X_1) \subseteq C \). One more application of ergodicity gives Theorem 8.2. We already observed that \( \approx_m \) is a countable Borel equivalence relation. (See Proposition 8.6). Hence there exists and some \( Z \subseteq X_1 \) with \( \mu(Z) > 0 \) and some group \( A \in C \) such that \( f(x) = A \) for each \( x \in Z \). Hence let \( M = \text{SL}_n(\mathbb{Z}) \cdot Z \). Since \( M \) is \( \text{SL}_n(\mathbb{Z}) \)-invariant, \( \mu(M) = 1 \) by ergodicity. Moreover \( f(M) \) is contained in the \( \approx_m \)-class of \( A \).

In the rest of this section we discuss the classification problem for finitely generated torsion-free abelian groups.
Definition 8.17. If $E_i, i < n$, where $n \leq \infty$, are Borel equivalence relations, with $E_i$ living on $X_i$, then we let $\bigoplus_{i<n} E_i$ be the equivalence relation on $\bigsqcup_{i<n} X_i = \bigcup_{i<n} X_i \times \{i\}$ given by given by

$$(x,j) \bigoplus_{i<n} E_i (y,k) \iff j = k \text{ and } x E_j y.$$ 

Theorem 8.18 (Thomas[42]). $\bigoplus_n \cong_n$ is not universal.

Proof. Let $S$ be a suitable countable group, and let $G = \text{SL}_3(\mathbb{Z}) \times S$. Consider the free action of $G$ on $((2)^G, \mu)$ where $\mu$ is the usual product probability measure. Suppose that $f: (2)^G \rightarrow \bigsqcup_n S(Q^n), x \mapsto A_x$ is a Borel reduction from $F(G, 2)$ to $\bigoplus_n \cong_n$. By ergodicity there exists a $G$-invariant subset $X_1 \subseteq (2)^G$ such that $f(X_1) \subseteq S(Q^n)$ for a fixed $n < \omega$. For ease of exposition we assume that $f: (2)^G \rightarrow S(Q^n)$. As in the proof of Theorem 8.2, the action of $\text{GL}_n(Q)$ is not free. Again, we get around this difficulty by considering the coarser equivalence relation of quasi-isomorphism.

For each $x \in (2)^G$, there are countably many possible values for $\text{QAut}(A_x) = \text{Stab}_{\text{GL}_n(Q)}([A_x])$. Then, by ergodicity there is a fixed subgroup $L \leq \text{GL}_n(Q)$ and some Borel $X_1 \subseteq (2)^G$, with $\mu(X_1) = 1$, such that $\text{QAut}(A_x) = L$ for all $x \in X_1$.

If $x, y \in (2)^G$ such that $x E_G y$, then there is $\varphi \in N_{\text{GL}_n(Q)}(L) = \{\psi \in \text{GL} : \psi L = L \psi\}$ such that $\varphi \cdot [A_x] = [A_y]$. So let $H = N/L$.

We can define a Borel cocycle $\alpha: G \times (2)^G \rightarrow H$ by setting $\alpha(g, x) \rightarrow H$ the unique $\overline{\varphi} \in H$ such that $\varphi \cdot [A_x] = [A_{g \cdot x}]$.

At this point we specify the nature of $S$. Let $S$ be a simple nonamenable countable group such that $S$ does not embed into any of the possible values for $H$. By Theorem 7.7, $\alpha$ is equivalent to a Borel homomorphism. We can assume that $\alpha: G \rightarrow H$ is a group homomorphism possibly deleting a null set. By the assumption on $S$, we have $S \leq \ker \alpha$. It follows that for every $g \in S$

$$[A_{g \cdot x}] = [A_x] \quad \mu\text{-a.e. } x \in (2)^G.$$ 

Therefore, up to a measure 0 set $f$ is a Borel homomorphism from $E_S$ to $\approx_n$. Now we recall the following result.

Theorem 8.19 (Losert-Rindler [27], Jones-Schmidt [21]). If $\Gamma \leq \Delta$ is a nonamenable group then the shift action $\Gamma \curvearrowright 2^\Delta$ is $E_0$-ergodic.

Proof. See [19, Theorem A4.1].
By Theorem 8.14 the quasi-equality relation $\approx_n$ is hyperfinite. It follows that there is a subset of $(2)^G$ of measure 1 that is mapped by $f$ into a single $\approx_n$-class. This is a contradiction. 

Notice that the following is an open question and an affirmative answer will imply an alternative prove of Theorem 8.18.

**Question 8.20.** Is $\sim_n$ essentially free, for $n \geq 2$?

### 9 The algebraic structure of bireducibility degrees

In this section we are concerned with the structural properties of countable Borel equivalence relations. We show that the class $C$ of countable Borel equivalence relations modulo Borel reducibility carries a rich additional algebraic structure.

**Definition 9.1** (Tarski [40]). A cardinal algebra is a triple $\langle A, +, \Sigma \rangle$ such that:

(I) $\langle A, + \rangle$ is an abelian monoid;

(II) $\Sigma: A^N \to A$ is an infinitary operation such that:

(i) $\sum_n a_n = a_0 + \sum_n a_{n+1}$;

(ii) $\sum_n (a_n + b_n) = \sum_n (a_n) + \sum_n (b_n)$

(iii) If $a + b = \sum_n c_n$, then there are $(a_n), (b_n) \in A^N$ with $a = \sum_n a_n$ and $b = \sum_n b_n$ such that $c_n = a_n + b_n$.

(iv) If $(a_n), (b_n) \in A^N$ with $a_n = b_n + a_{n+1}$, then there is $c$ such that for each $n$

$$a_n = c + \sum_i b_{n+i}.$$ 

Cardinal algebras were introduced by Tarski to axiomatize addition of cardinals without in absence of the choice axiom.

**Remark 9.2.** When $\langle A, +, \Sigma \rangle$ is a cardinal algebra, we can define a partial order on $A$ by declaring $a \leq b$ if and only if $\exists c (a + c = b)$.

Given Borel equivalence relations $E, F$ on standard Borel spaces $X$ and $Y$ respectively, we say that $E$ invarianlty embeds in to $F$ (written $E \sqsubseteq_B F$) if there is a Borel $F$-invariant $A \subseteq Y$ so that $E \equiv_B F \upharpoonright A$. 

40
Definition 9.3 (Kechris-McDonald [24]). A class of Borel equivalence relation is Tarskian if and only if

1. $\emptyset \in E$.
2. If $F \in E$ and $E \sqsubseteq_B F$, then $E \in E$.
3. If $F_0, F_1, \ldots \in E$, then $\bigoplus_n F_n \in E$.
4. If $E, F_0, F_1, \ldots \in E$ and $E \sim_B \bigoplus_n F_n$, then there are $E_n \in E$ with $F_n \sim_B E_n$ such that $E \equiv_B \bigoplus_n E_n$.
5. $E, F \in E$ live on $X$ and $Y$, respectively. Then $f : X \to Y$, $E \leq_B F$. Then the saturation $A = [f(X)]_F$ is a Borel subset of $Y$ and $E \sim_B F \upharpoonright A$.

If $\mathcal{E}$ is a class of Borel equivalence relations, let $[\mathcal{E}] = \{[E] : E \in \mathcal{E}\}$, where $[E] = \{F \in \mathcal{E} : E \sim_B F\}$. Then if $E, F \in \mathcal{E}$ let $[E] + [F] = [E \oplus F]$. Moreover, if $F_n \in \mathcal{E}$, let $\sum_n [F_n] = [\bigoplus_n F_n]$.

Exercise 9.4. If $\mathcal{E}$ is a Tarskian class of Borel equivalence relation then $\langle [\mathcal{E}], +, \sum \rangle$ is a cardinal algebra. Moreover, for $E, F \in \mathcal{E}$ we have that $E \leq_B F \iff [E] \leq [F]$.

Corollary 9.5. $\langle [\mathcal{C}], +, \sum \rangle$ is a cardinal algebra.

Let $\mathcal{B}$ be the class of all Borel equivalence relations.

Question 9.6. Is $\langle [\mathcal{B}], +, \sum \rangle$ a cardinal algebra?

Remark 9.7. For $E, F \in \mathcal{B}$ it does NOT hold in general that $E \leq_B F \iff [E] \leq [F]$.

Theorem 9.8 (Hjorth [17]). There is a Borel equivalence relation $E$ that Borel reduces to some CBER $F$ but is not Borel equivalent to any CBER.

It follows that $[E] \not\leq [F]$. For suppose that $[E] \leq [F]$ and let $H$ be such that $E \oplus H \sim_B F$. Let $f : Y \to X \sqcup Z$ be a Borel reduction from $F$ to $E \oplus H$. Then $A = f^{-1}(X)$ would be an $F$-invariant subset of $Y$, and $E \sim_B F \upharpoonright A$, which contradicts Theorem 9.8.

Corollary 9.5 has several consequences on the structure of $C$.

(A) Existence of suprema. Any increasing sequence $F_0, F_1, \ldots$ admits a least upper bound in $\leq_B$.
(B) Interpolation. (Interpolation) If \( S, T \) are countable sets of countable Borel equivalence relations and every \( E \in S \) is Borel reducible to each \( F \in T \), then there is a countable Borel equivalence relation \( F_0 \) such that for all \( E \in S, F \in T \)

\[ E \leq_B F_0 \leq_B F. \]

(C) Cancellation. If \( n > 0 \) and \( E, F \) are countable Borel equivalence relations, then

\[ nE \leq_B nF \implies E \leq_B F, \]

thus

\[ nE \sim_B nF \implies E \sim_B F. \]

**Definition 9.9.** If \( E \) and \( F \) are Borel equivalence relations on \( X \) and \( Y \) respectively, then the product \( E \times F \) is an equivalence relation on \( X \times Y \). Precisely,

\[ (x, y) \in E \times F (x', y') \iff x E x' \text{ and } y F y'. \]

If \( E \) is a Borel equivalence relations on \( X \), we denote by \( E^n = X \times \cdots \times X \). Notice that \( \langle [C], \cdot \rangle \)

is an abelian monoid with \( [E] \cdot [F] = [E \times F] \) and identity element \([=1]\), the equality relation on a space of one element.

**Theorem 9.10** (Kechris-Macdonald [24]). There are CBERs \( E \prec_B F \) such that \( E^2 \sim_B F^2 \).

**Lemma 9.11.** Suppose that \( R, S \) are Borel equivalence relations on \( (X, \mu) \) and \( (Y, \nu) \) respectively such that

(i) \( R \) and \( S \) have infinitely many equivalence classes;

(ii) \( R^2 \sim_B R \) and \( S^2 \sim_B S \)

(iii) \( R \) and \( S \) are \( \mu \)-ergodic and \( \nu \)-ergodic respectively, and for all \( R \)-invariant Borel \( A \subseteq X \), with \( \mu(A) = 1 \), we have \( R \upharpoonright A \preceq_B S \). (Similarly, for all \( S \)-invariant Borel \( C \subseteq Y \), with \( \mu(A) = 1 \), \( S \upharpoonright C \preceq_B R \).)

Then, if \( E = R \oplus S \) and \( F = R \times S \), \( E \prec_B \) but \( E^2 \sim_B F^2 \).

**Proof.** First, we prove that \( E \leq_B F \). It follows by Exercise 3.2 that for fixed \( x_0 \in X \) and \( y_0 \in Y \),

\[ E = R \oplus S \leq_B R \upharpoonright (X \setminus \{x_0\}) \oplus S \upharpoonright (Y \setminus \{y_0\}). \]
Let $Z = (X \setminus [x_0]_R) \sqcup (Y \setminus [y_0]_S)$. Then the function $f : Z \to X \times Y, x \mapsto (x, y_0), y \mapsto (x_0, y)$ is a Borel reduction from the latter equivalence relation in (1) to $R \times S = F$.

Next, we prove that $E^2 \sim_B F^2$. Since $F^2 = (R \times S)^2 \sim_B R^2 \times S^2 \sim_B R \times S$, it suffices to show that $E^2 = (R \oplus S)^2 \sim_B R \times S$.

Last, we show that $E <_B F$. Suppose that $f : X \times Y \to X \sqcup Y$ is a Borel reduction from $F = R \times S$ to $E = R \oplus S$. Let $X_0 = f^{-1}(X)$ and $Y_0 = f^{-1}(Y)$ so that $X_0 \sqcup Y_0 = X \times Y$.

Now notice that both $X_0$ and $Y_0$ are $R \times S$-invariant.

Claim. $R \times S$ is $\mu \times \nu$-ergodic.

It follows that either $\mu \times \nu(X_0) = 1$ or $\mu \times \nu(Y_0) = 0$ (which implies $\mu \times \nu(Y_0) = 1$). In the first case, there is $x \in X$ such that $\nu((X_0)_x) = 1$. Also note that the vertical section $(X_0)_x$ is $S$-invariant. Then the function

$$(X_0)_x \to X \sqcup Y$$

$$y \mapsto f(x, y)(\in X)$$

witnesses that $S \upharpoonright (X_0)_x \leq_B R$. This is contradictory with (iii). If $\mu \times \nu(Y_0) = 1$, we argue in a similar way. \qed

Proof of Theorem 9.10. It suffices to find two countable Borel equivalence relations $R$ are $S$ that satisfy conditions (i)–(iii) from Lemma 9.11. The proof of this fact uses the following result.

Theorem 9.12 (essentially Ol’shanski [35]). There is a countable Kazhdan, torsion free, simple group $\Gamma$, and a countably infinite Kazhdan, torsion, simple group $\Delta$.

Given $\Gamma$ and $\Delta$ as above, define $\Gamma^* = \Gamma \oplus \Gamma \oplus \cdots$ and $\Delta^* = \Gamma \oplus \Gamma \oplus \cdots$. Notice that any homomorphism $\Gamma \to \Delta^*$ or $\Delta \to \Gamma^*$ is trivial. Now put $R = F(\Gamma^*, [0, 1])$ and $S = F(\Delta^*, [0, 1])$ and let $\mu$ and $\nu$ the usual product measure on $([0, 1])^{\Gamma^*}$ and $([0, 1])^{\Delta^*}$, respectively. It is clear that (i) holds.

To see (ii) it is enough to show that $R^2$. Notice that $R^2$ coincide with the equivalence relation on $([0, 1])^{\Gamma^*} \times ([0, 1])^{\Gamma^*}$ induced by the free action of $\Gamma^* \times \Gamma^*$ defined by $(\gamma, \delta) \cdot (x, y) = (\gamma \cdot x, \delta \cdot y)$. Since all uncountable standard Borel space are isomorphic, there is a free action $a : \Gamma^* \times \Gamma^* \simeq [0, 1]$ inducing a countable Borel equivalence relation $E_a \cong R^2$. 43
It is a general fact that if a countable group $G \curvearrowright X$ freely, then the map $X \to X^G, x \mapsto (g \mapsto g^{-1} \cdot z)$ witness that $E^X_G \leq_B F(G, [0, 1])$. So we have

$$R^2 \cong_B E_a \leq_B F(\Gamma^* \times \Gamma^*, [0, 1]) \cong_B F(\Gamma^*, [0, 1]) = R.$$ 

Hence $R \sim_B R^2$ and similarly $S \sim_B S^2$.

For (iii) notice that $R$ and $S$ are $\mu$-ergodic and $\nu$-ergodic respectively. It remains to show that if $A \subseteq 2^\mathbb{N}$ is $R$-invariant and $\mu(A) = 1$, then $R | A \not\leq_B S$.

Suppose that $f : A \to Y$ witnesses $R | A \leq_B S$. Then for every $g \in \Gamma, x \in A$ there is unique $a(g, x) \in \Delta^*$ such that $f(g \cdot a) = a(g, x) f(x)$. Thus the function $a : G \times X \to \Delta^*$ is a Borel cocycle. By Popa’s superrigidity theorem (see Theorem 7.7) there is a (necessarily trivial) group homomorphism $h : \Gamma \to \Delta^*$ and a Borel reduction $b : X \to \Delta^*$ such that

$$a(g, x) = b(g \cdot x)(g \cdot x) h(x) b(x)^{-1} \mu - a.e. \ x \in A.$$ 

Now consider the function $f'(x) = b(x)^{-1} f(x)$.

Notice that, up to a null set, $f'$ is Borel reduction from $R | A$ to $S$. Moreover, it is not hard to see that $f'$ is $G$-invariant (i.e., $f'(g \cdot x) = f'(x)$). By ergodicity of the shift action of $\Gamma \curvearrowright ([0, 1]^{\Gamma}, \mu)$, it follows that $f'$ is constant on some $X_0 \subseteq A$ with $\mu(X_0) = 1$. Clearly the latter condition is incompatible with $f'$ being a Borel reduction, because Borel reductions are one-to-one.

\[ \square \]

### 10 Martin’s conjecture

In this section we discuss a crucial interplay between recursion theory and descriptive set theory.

**Definition 10.1.** For $r, s \in 2^\mathbb{N}$ we say that $r$ is Turing reducible (written $r \leq_T s$) if and only if there exists an oracle Turing machine that computes $r$ when its oracle tape contains $s$.

We say that $r$ and $s$ are Turing equivalent (written $r \equiv_T s$) iff $r \leq_T s$ and $s \leq_T r$.

It is clear from the definition that $\equiv_T$ is a countable Borel equivalence relation. In fact, every point $s \in 2^\mathbb{N}$ has countably many $\leq_T$-predecessors.

**Definition 10.2.** For each $r \in 2^\mathbb{N}$, the corresponding cone is the Borel subset $C \subseteq 2^\mathbb{N}$ defined by

$$C = \{ s \in 2^\mathbb{N} \mid r \leq_T s \}.$$ 

**Theorem 10.3** (Martin). If $A \subseteq 2^\mathbb{N}$ is a $\equiv_T$-invariant Borel subset, then either $A$ contains a cone or $2^\mathbb{N} \setminus A$ contains a cone.
Proof. The proof uses Martin’s result that every Borel game is determined. Let \( A \subseteq 2^\mathbb{N} \) be a Borel \( \equiv_T \)-invariant set. Consider the Gale-Stewart game \( G(A) \) with payoff set \( A \). That is, the game described by the following plays.

<table>
<thead>
<tr>
<th></th>
<th>( s_0 \in {0, 1} )</th>
<th>( s_2 \in {0, 1} )</th>
<th>( \cdots )</th>
<th>( s_{2n} \in {0, 1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>( s_1 \in {0, 1} )</td>
<td>( \cdots )</td>
<td>( s_{2n+1} \in {0, 1} )</td>
<td></td>
</tr>
</tbody>
</table>

where Player I wins if and only if \( s = (s_n)_{n \in \mathbb{N}} \in A \). By Borel determinacy theorem, one of the two players has a winner strategy.

If \( \sigma : 2^{<\mathbb{N}} \to 2 \) is a winning strategy for Player I, then consider the cone \( C = \{ t \in 2^\mathbb{N} : \sigma \leq_T t \} \). It follows that \( C \subseteq A \). To see this, let \( t \in C \) so we have \( \sigma \leq_T t \). Now consider the following plays for \( G(A) \)

- Player I plays some \( s_0 \in \{0, 1\} \), and then follows the winning strategy \( \sigma \) (i.e., \( s_{2n} = \sigma(s_0 \wedge \cdots \wedge s_{2n-1}) \));
- Player II plays \( s_{2n+1} = t_n \).

Since \( \sigma \) is a winning strategy for Player I, it follows that \( s \in A \). Moreover, it is clear that \( t \leq_T s \).

On the other hand, we have \( \sigma \leq_T t \) and \( s \leq \sigma \oplus t \), which implies that \( s \leq_T t \). Since \( A \) is \( \equiv_T \)-invariant, we have \( t \in A \). \( \square \)

**Corollary 10.4.** If \( f : 2^\mathbb{N} \to 2^\mathbb{N} \) is a Borel function such that \( x \equiv_T y \) implies \( f(x) = f(y) \), then there is a cone \( C \subseteq 2^\mathbb{N} \) such that \( f \upharpoonright C \) is constant.

**Proof.** It follows by Theorem 10.3 that for each \( n \in \mathbb{N} \), either \( \{ x \in 2^\mathbb{N} : f(x)_n = 0 \} \) or \( \{ x \in 2^\mathbb{N} : f(x)_n = 1 \} \) contains a cone. So, for each \( n \in \mathbb{N} \), there is \( \epsilon_n \in \{0, 1\} \) such that \( X_n = \{ x \in 2^\mathbb{N} : f(x)_n = \epsilon_n \} \) contains a cone \( C_n \). Let \( D = \cap_n C_n \). Clearly \( f \upharpoonright D \) is constant. Moreover, we notice that \( \cap_n C_n \) contains a cone. To see this let \( r_n \in 2^\mathbb{N} \) be such that \( C_n = \{ s \in 2^\mathbb{N} : r_n \leq_T s \} \). Then let \( \bigoplus_n r_n \) be the join of the \( r_n \)'s and let \( D = \cap_n C_n \). Clearly, \( r_n \leq_T \bigoplus_n r_n \). It follows that \( C = \{ s \in 2^\mathbb{N} : \bigoplus_n r_n \leq_T s \} \subseteq D \). \( \square \)

We say that \( A \subseteq 2^\mathbb{N} \) is \( \leq_T \)-cofinal if for every \( s \in 2^\mathbb{N} \) there is \( s' \in A \) such that \( s \leq_T s' \).

**Exercise 10.5.** Let \( A \subseteq 2^\mathbb{N} \) be a \( \equiv_T \)-invariant Borel subset. If \( A \) is \( \leq_T \)-cofinal then \( A \) contains a cone.

**Definition 10.6.** A Turing invariant function is a Borel homomorphism \( f : 2^\mathbb{N} \to 2^\mathbb{N} \) from \( \equiv_T \) to itself.

The following open problem is the Borel version of the first part of a conjecture of Martin [30].
**Conjecture 10.7 (MC).** If $f: 2^\mathbb{N} \to 2^\mathbb{N}$ is a Turing invariant Borel function, then exactly one of the following holds:

(I) There is a cone $C \subseteq 2^\mathbb{N}$, $f$ maps $C$ into a single $\equiv_T$-class;

(II) There is a cone $C \subseteq 2^\mathbb{N}$ such that $x \leq_T f(x)$ for all $x \in C$.

**Remark 10.8.** Without the requirement that $f$ be Borel the statement above is false in ZFC. In fact, the axiom of choice implies the existence of several pathological Turing invariant functions. The original version of Martin’s conjecture\(^2\) is formulated in ZF + DC + AD. (See also the excellent introduction of [31].)

The only known progress towards MC is the following result by Slaman and Steel.

**Theorem 10.9 (Slaman-Steel [39]).** Suppose that $f: 2^\mathbb{N} \to 2^\mathbb{N}$ is a Turing invariant Borel function. If there is a cone $C \subseteq 2^\mathbb{N}$ such that $f(x) <_T x$ for all $x \in C$, then there is a cone $D \subseteq C$ such that $f$ maps $D$ into a single $\equiv_T$-class.

(MC\('\)) If $f: 2^\mathbb{N} \to 2^\mathbb{N}$ is a Turing invariant Borel function, then exactly one of the following holds:

(I) For all $x \in 2^\mathbb{N}$, there is $y$ such that $f(y) <_T y$.

(II) For all $x \in 2^\mathbb{N}$, there is $y$ such that $y \leq_T f(y)$.

The following observation will be useful later in this section.

**Exercise 10.10.** (MC) is equivalent to (MC\('\))

**The Borel complexity of $\equiv_T$**

In this subsection we investigate some consequences of MC. First it is natural to ask what is the Borel complexity of $\equiv_T$.

**Theorem 10.11 (MC).** $(\equiv_T)^2 \not\leq_B \equiv_T$. Thus, $\equiv_T$ is not universal.

**Proof.** Suppose that $f: 2^\mathbb{N} \times 2^\mathbb{N} \to 2^\mathbb{N}$ is a Borel reduction from $(\equiv_T)^2$ to $\equiv_T$. Let $r, s \in 2^\mathbb{N}$ be such that $r \not\equiv_T t$. Define $f_0: 2^\mathbb{N} \to 2^\mathbb{N}, s \mapsto f(r, s)$ and $f_1: 2^\mathbb{N} \to 2^\mathbb{N}, s \mapsto f(t, s)$. Since $r$ and $t$ are not $\equiv_T$-equivalent, it is clear that $A_0 = [f_0(s^\mathbb{N})]$ and $A_1 = [f_1(2^\mathbb{N})]$ are disjoint $\equiv_T$-invariant Borel set.

\(^2\)Martin’s conjecture is one of the few unsolved Victoria Delfino’s problem. More on the character of Victoria Delfino and the list of problems entitled to her can be found in [5].
Also, for each $i \in \{0, 1\}$, it cannot be the case that $f_i$ is constant on a cone. Hence, there is a cone $C_i \subseteq 2^N$ such that $x \leq_T f_i(x)$ for all $x \in C_i$. This implies that each $A_i$ is $\leq_T$-cofinal, then each $A_i$ contains a cone by Exercise 10.5. This contradicts the fact that $A_0$ and $A_1$ are disjoint. \hfill \square

Although the exact Borel complexity of $\equiv_T$ is unknown. One can prove that $\equiv_T$ is universal with respect to weak Borel reducibility. (cf. Definition 7.22.)

**Definition 10.12.** A countable Borel equivalence relation $F$ is *weakly universal* if every countable Borel equivalence relation $E$ is weakly Borel reducible to $F$ (in symbols, $E \leq^w_B F$).

The free group $F_2$ can be identified with a suitable group of recursive permutations of $\N$. So the orbit equivalence relation $E(F_2, 2)$ is contained in $\equiv_T$. This implies that $\equiv_T$ is weakly universal.

**Theorem 10.13 (MC).** If $f : 2^N \to 2^N$ is a Borel $\equiv_T$-invariant function, then either

(i) There is a cone $C \subseteq 2^N$ such that $f$ maps $C$ into a single $\equiv_T$-class;

(ii) There is a cone $C \subseteq 2^N$ such that $f \mid C$ is a weak Borel reduction from $\equiv_T \mid C$ to $\equiv_T$. Moreover, for every cone $D \subseteq 2^N$ the saturation $[f(D)]_{\equiv_T}$ contains a cone.

**Proof.** If (i) fails, then (MC) implies the existence of $C \subseteq 2^N$ such that $x \leq_T f(x)$ for all $x \in C$. Since every $x \in 2^N$ has countably many predecessors $f \mid C$ is necessarily countable-to-one. So, $f \mid C$ is a weak Borel reduction from $\equiv_T \mid C$ to $\equiv_T$. Moreover, let $D \subseteq 2^N$ be any cone and let $D_0 = D \cap C$. Since $f \mid C$ is countable-to-one, it follows that $f(D_0)$ is Borel. (See Exercise 1.16.) Therefore, the saturation $[f(D_0)]_{\equiv_T}$ is a Borel $\equiv_T$-invariant set. Moreover, $[f(D_0)]$ is $\leq_T$-cofinal. It follows by Exercise 10.3 that $[f(D_0)]_{\equiv_T}$ contains a cone. \hfill \square

**Corollary 10.14 (MC).** If $A \subseteq 2^N$ is a $\equiv_T$-invariant Borel subset, then $\equiv_T \mid A$ is weakly universal if and only if $A$ contains a cone.

**Proof.** If $\equiv_T \mid A$ is weakly universal, then there is a weak Borel reduction $h : 2^N \to A$ from $\equiv_T$ to $\equiv_T \mid A$. Then condition (i) of Theorem 10.13 must hold and $[h(2^N)]_{\equiv_T}$ contains a cone.

Conversely, suppose that the cone $C = \{s \in 2^N : r \leq s\}$ is contained in $A$. Given two reals $x, y \in 2^N$, the join $x \oplus y$ is the real $x \oplus y$ defined $(x \oplus y)(2n) = x(n)$ and $(x \oplus y)(2n + 1) = x(n)$ for all $n \in \N$. Then, the map $s \mapsto s \oplus r$ is an injective weakly Borel reduction from $\equiv_T$ to $\equiv_T \mid A$. \hfill \square
**Ergodicity** for Turing equivalence

**Definition 10.15.** Let \( E \) be a countable Borel equivalence relation on a countable Borel space \( X \). We say that \( \equiv_T \) is \( E \)-\textit{m-ergodic} if for every Borel homomorphism \( f : 2^\mathbb{N} \to X \) from \( \equiv_T \) to \( E \), there is a cone \( C \subseteq 2^\mathbb{N} \) such that \( f \) maps \( C \) into a single \( E \)-class.

Notice that Corollary 10.4 can be reformulated by saying that \( \equiv_T \) is \( \text{id}(2^\mathbb{N}) \)-\textit{m-ergodic}. Currently, we do not know any example of nonsmooth countable Borel equivalence relation \( E \) such that \( \equiv_T \) is \( E \)-\textit{m-ergodic}. In particular the following question is open.

**Question 10.16.** Is \( \equiv_T \) \( E_0 \)-\textit{m-ergodic}?

**Theorem 10.17 (MC).** If \( E \) is a countable Borel equivalence relation then either one of the following is true:

(a) \( E \) is weakly universal; or

(b) \( \equiv_T \) is \( E \)-\textit{m-ergodic}.

**Proof.** If \( E \) is weakly universal, then there is a countable-to-one Borel homomorphism from \( \equiv_T \) to \( E \). Hence (b) is false.

Next, suppose that (b) fails. Then, there is a Borel homomorphism \( \varphi : 2^\mathbb{N} \to X \) from \( \equiv_T \) to \( E \) such that for every cone \( C \subseteq 2^\mathbb{N} \), \( \varphi(C) \) is not contained into a single \( E \)-class. Since \( \equiv_T \) is weakly universal, there is a weak Borel reduction \( \psi : X \to 2^\mathbb{N} \) from \( E \) to \( \equiv_T \). Then \( \theta = \psi \circ \varphi \) is a Borel \( \equiv_T \)-invariant map from \( \equiv_T \) to \( \equiv_T \). Thus, Martin’s conjecture implies that \( \theta \) satisfies either (i) or (ii) of Theorem 10.13.

We claim that (i) cannot hold. To see this, suppose that there is a cone \( C_0 \subseteq 2^\mathbb{N} \) such that \( \theta \) maps \( C \) into a single class \([x]_{\equiv_T}\). Then, \( Y = \psi^{-1}([x]_{\equiv_T}) \) is countable because \( \psi \) is countable-to-one, and \( \varphi \) maps \( C_0 \) into a \( Y \). Notice that there is \( y \in Y \) such that \( Z = \varphi^{-1}(y) \) is a \( \leq_T \)-cofinal Borel subset of \( 2^\mathbb{N} \). (Why? Exercise.) Then, Exercise 10.5 implies that the saturation \([Z]_{\equiv_T}\) contains a cone \( D \). It follows that \( D \) is mapped by \( \varphi \) into a single \( E \)-class, which contradicts the assumption that \( \varphi \) witnesses the failure of (b).

So, Theorem 10.13 (ii) implies that there is a cone \( C \subseteq 2^\mathbb{N} \) such that \( \theta \upharpoonright C \) is a weak Borel reduction from \( \equiv_T \upharpoonright C \) to \( \equiv_T \). In particular, \( \theta \upharpoonright C \) is a countable-to-one map, and \( \varphi \upharpoonright C \) is necessarily countable-to-one. We conclude that \( (\equiv_T \upharpoonright C) \leq_B^w E \), which implies that \( E \) is universal as desired. \( \square \)
**Exercise 10.18** (MC). Let $E$, $F$ be countable Borel equivalence relations on $X$, $Y$, respectively. Suppose that $E$ is weakly universal and that $F$ is not weakly universal. If $\varphi : X \to Y$ is a Borel homomorphism from $E$ to $F$, then there exists a Borel $X_0 \subseteq X$ such that:

(i) $E \upharpoonright X_0$ is weakly universal

(ii) $\varphi$ maps $X_0$ into a single $F$-class.

**Definition 10.19.** Let $G$ be a countable group. We say that $G$ is **weakly action universal** if there is a Borel $G$-action on some standard Borel space $X$ such that $E^X_G$ is weakly universal.

**Definition 10.20.** Let $\approx_G$ be the **conjugacy equivalence relation** on $S(G) = \{H \subseteq G : H \leq G\}$, the standard Borel space of subgroups of $G$. That is, given $H, K \leq G$, we say that

$H \approx_G K \iff \exists g \in G (K = gHg^{-1})$.

**Theorem 10.21** (Thomas [44]). (MC) If $G$ is a countable group then the following are equivalent:

(i) $G$ is weakly action universal;

(ii) $\approx_G$ is weakly universal.

**Proof.** Clearly, (ii) implies (i). Suppose that $G \to X$, and $E^X_G$ is weakly universal and $\approx_G$ is not weakly universal.

Consider the stabilizer Borel map $\sigma : X \to S(G)$, $\sigma(x) = G_x = \{g \in G : g \cdot x = x\}$. Notice that $\sigma$ is a Borel homomorphism from $E^X_G$ to $\approx_G$ as, whenever $g \cdot x = y$, then $G_y = g \cdot G_x g^{-1}$. Now, since $E^X_G$ is weakly universal, there exists a weak Borel reduction $\varphi : 2^N \to X$ from $\equiv_T$ to $E^X_G$. So $\theta = \sigma \circ \varphi$ is a Borel homomorphism from $\equiv_T$ to $\approx_G$. It follows from Theorem 10.17 that there is a cone $C \subseteq 2^N$ such that $\theta(C)$ is contained into a single $\approx_G$-class. Up to redefining $\varphi$, we can assume that there is a fixed $K \leq G$ such that $G_{\varphi(x)} = K$ for each $x \in C$. Let $X_0 = \{x \in X : G_{x_0} = K\}$. Therefore, we have $\equiv_T \upharpoonright C \leq_{\approx_G} E^X_G \upharpoonright X_0$, and this implies that $E^X_G \upharpoonright X_0$ is weakly Borel universal because so is $\equiv_T \upharpoonright C$. (Cf. Corollary 10.14.)

Notice that for all $x, y \in X_0$, whenever $g \cdot x = y$ for some $g \in G$, then $g \in N_G(K) = \{g \in G : Kg = gK\}$. Then $E^X_G \upharpoonright X_0$ is the equivalence relation induced by the obvious free action of $N_G(K)/K$ on $X_0$. It follows by Theorem 7.24 that $E^X_G \upharpoonright X_0$ is not weakly universal. A contradiction. \qed
Uncountably many weakly universal CBERs

**Theorem 10.22** (Thomas [43]). (MC) There exist uncountable many weakly universal countable Borel equivalence relations up to Borel bi-reducibility.

**Definition 10.23.** Let $E, F$ be equivalence relations on the standard Borel spaces $X, Y$, respectively. Suppose that $\mu$ is an $E$-invariant probability measure on $X$. Then $E$ is $F$-ergodic if and only if for every Borel homomorphism $f : X \to Y$ from $E$ to $F$, there is a Borel $Z \subseteq X$ with $\mu(Z) = 1$ such that $f$ maps $Z$ into a single $F$-class.

Notice that ergodicity defined as in Definition 10.23 transfers to the measure theoretical context.

**Exercise 10.24.** If $E$ is $F$-ergodic then for every $\mu$-measurable homomorphism from $E$ to $F$, there is a Borel $Z \subseteq X$ with $\mu(Z) = 1$ such that $h$ maps $Z$ into a single $F$-class.

**Fact 10.25.** There is an uncountable family $G = \{G_\alpha : \alpha \in 2^\mathbb{N}\}$ of finitely generated groups such that:

(a) $\text{SL}_3(\mathbb{Z}) \triangleleft G_\alpha$;

(b) $G_\alpha$ has no nontrivial finite normal subgroups;

(c) if $\alpha \neq \beta$ then $G_\alpha$ does not embed into $G_\beta$.

Moreover, we can assume that each of the $G_\alpha$ is a group on $\mathbb{N}$ and that $\{G_\alpha : \alpha \in 2^\mathbb{N}\}$ forms a Borel of the space of countable groups.

In the sequel of this section let $X_\alpha := (2)^{G_\alpha}$ and $F_\alpha := F(G_\alpha, 2)$.

As a consequence of Popa’s superrigidity theorem we easily obtain the following:

**Fact 10.26.** If $\alpha \neq \beta$, then $F_\alpha$ is $F_\beta$-ergodic with respect to the product probability measure on $X_\alpha$.

An important consequence of the ergodicity result above is that all $F_\alpha$ are not weakly universal. It follows that $\equiv_T$ is $F_\alpha$-m-ergodic, for each $\alpha \in 2^\mathbb{N}$ by Theorem 10.17.

Now we are ready to prove Theorem 10.22, which follows from the following result.

**Theorem 10.27** (MC). If $\alpha \neq \beta$, then $(\equiv_T \times F_\alpha)$ is not Borel reducible to $(\equiv_T \times F_\beta)$.

In proof we shall use a consistency result of Martin and Solovay [32].

**Theorem 10.28** ([32]). If $(MA) + 2^{\aleph_0} > \aleph_1$, then every $\Sigma^1_2$ set is measurable.
Proof of Theorem 10.27. Suppose that $f : 2^\mathbb{N} \times \alpha \to 2^\mathbb{N} \times \beta$ is a Borel reduction from $(\equiv_T \times F_\alpha)$ to $(\equiv_T \times F_\beta)$. Let $\varphi : 2^\mathbb{N} \times \alpha \to 2^\mathbb{N}$ and $\psi : 2^\mathbb{N} \times \alpha \to \beta$ be the Borel function such that $f(r, x) = (\varphi(r, x), \psi(r, x))$ for all $r \in 2^\mathbb{N}, x \in \alpha$. For each $x \in \alpha$, we let

$$\psi_x : 2^\mathbb{N} \to \beta, r \mapsto \psi(r, x).$$

Notice that $r \equiv_T$ implies $\psi_x(r) F_\beta \psi_x(s)$, so $\psi_x$ gives a Borel homomorphism from $\equiv_T$ to $F_\beta$ for any $x \in \alpha$. Since $\equiv_T$ is $F_\beta$-ergodic, for each $x \in \alpha$, there is a cone $C_x \subseteq \alpha$ such that $\psi_x$ maps $C_x$ into a single $F_\beta$-class $d_x$.

Goal. Define a Borel homomorphism from $F_\alpha$ to $F_\beta$ and use the fact that $F_\alpha$ is $F_\beta$-ergodic.

It is unclear whether such Borel map exists, but fortunately we can work in the more general measure theoretical context and use some metamathematical tricks to get around this difficulty.

First observe that if $x F_\alpha y$ then $d_x = d_y$. To see that pick some $r \in C_y$. Notice that if $x F_\alpha y$, then $(r, x) (\equiv_T \times E_\alpha) (r, y)$, which implies that $\psi_x(r) = \psi_y(r) \in d_y$. Thus, it suffices to define a function $h : \alpha \to \beta$ such that $h(x) \in d_x$, for all $x \in \alpha$.

Consider the following relation:

$$(x, z) \in R \iff \exists s \forall r (s \leq_T r \implies \psi(r, x) F_\beta z).$$

It is immediate by the definition that $R$ is a $\Sigma^1_2$ subset of $\alpha \times \beta$.

By a classical result of Kondô [22, Corollary 38.7], there exists a uniformizing function $h$ for $R$. Observe that whenever $r \in C_x$, we have $d_x = [\psi(r, x)]_{F_\beta}$. Thus, the uniformization $h$ is a $\Sigma^1_2$ homomorphism from $F_\alpha$ to $F_\beta$.

There is no reason to think that such $h$ would be Borel, however if we move to a suitable forcing extension $V^P$ in which Martin’s axiom (MA) and $2^{\aleph_0} > \aleph_1$ hold, then $V^P$ proves that $h$ is $\mu$-measurable. Hence we can work in $V^P$ and use the measure theoretical version of ergodicity (see Exercise 10.24). Since $F_\alpha$ is $F_\beta$-ergodic, there is $Z \subseteq \alpha$ Borel, with $\mu(Z) = 1$, such that $h$ maps $F_\beta$ into a single $F_\beta$-class $c$.

Now fix some $z \in Z$, and consider the map $\varphi_x : 2^\mathbb{N} \to 2^\mathbb{N}, r \mapsto \varphi(r, x)$. It is clear that $\varphi_x$ is a Borel homomorphism from $\equiv_T$ to $\equiv_T$. Moreover, it cannot be the case that $\varphi_x$ is constant on a cone. In fact, the restriction $\varphi_x \upharpoonright C_x$ is a Borel reduction from $\equiv_T \upharpoonright C_x$ to $\equiv_T$. It follows from Theorem 10.13 (ii) that the saturation $[\varphi_x(C_x)]_{F_\beta}$ contains a cone $D_z$. 

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So, if \( x, y \in \mathbb{Z} \) are such that \( x \not\sim_{F_{\alpha}} y \), then we can fined some \( r \in C_x \) and \( s \in C_y \) such that \( \psi_x(r) \equiv_T \psi_x(s) \). Therefore

\[
f(r, x) \equiv_T F_{\beta} f(s, y),
\]

which contradicts with the fact that \( f \) is a Borel reduction. To sum up, we have \( V^P \models "(\equiv_T \times F_{\alpha}) \leq_B (\equiv_T \times F_{\beta})" \). That statement is a \( \Pi^1_2 \) property of \( \alpha \) and \( \beta \). So, Shoenfield’s absoluteness theorem implies that \( (\equiv_T \times F_{\alpha}) \leq_B (\equiv_T \times F_{\beta}) \) in the ground model. (See the appendix A.) A crucial point of this proof is that \( (MC) \) is equivalent to \( (MC') \), which is therefore absolute. So another application of Shoenfield’s absoluteness ensures that \( V^P \models (MC') \).

\[ \square \]

## A Reductions and absoluteness

Let \( V \) be a fixed base universe of set theory and let \( \mathbb{P} \) be a notion of forcing. Then we will write \( V^P \) for the corresponding generic extension when we do not wish to specify the generic filter \( G \subseteq \mathbb{P} \). If \( R \) is a projective relation on the Polish space \( X \), then \( X^{V^P}, R^{V^P} \) will denote the sets obtained by applying the definitions of \( X, R \) within \( V^P \). In particular, suppose that \( E \) is a Borel equivalence relation on the Polish space \( X \). Then the Shoenfield Absoluteness Theorem [? Theorem 25.20] implies that \( X^{V^P} \cap V = X \) and \( E^{V^P} \cap V = E \), that \( E^{V^P} \) is a Borel equivalence relation on \( X^{V^P} \), and that the following result holds.

**Theorem A.1.** If \( E, F \) are Borel equivalence relations on the Polish spaces \( X, Y \) and \( \theta : X \to Y \) is a Borel reduction from \( E \) to \( F \), then \( \theta^{V^P} \) is a Borel reduction from \( E^{V^P} \) to \( F^{V^P} \).

**Remark A.2.** The analogue of the result above for the \( \Sigma^1_1 \) equivalence relations is also true.
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